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DEFECTS, DUALITIES AND THE GEOMETRY OF STRINGS VIA GERBES II. GENERALISED GEOMETRIES WITH A TWIST, THE GAUGE ANOMALY AND THE GAUGE-SYMMETRY DEFECT

RAFAŁ R. SUSZEK*

Abstract. This is the second in a series of papers discussing, in the framework of gerbe theory, canonical and geometric aspects of the two-dimensional non-linear sigma model in the presence of conformal defects in the world-sheet. Employing the formal tools worked out in the first paper of the series, 1101.1126 [hep-th], a thorough analysis of rigid symmetries of the sigma model is carried out, with emphasis on algebraic structures on generalised tangent bundles over the target space of the theory and over its state space that give rise to a realisation of the symmetry algebra on states. The analysis leads to a proposal for a novel differential-algebraic construct extending the original definition of the (gerbe-twisted) Courant algebroid on the generalised tangent bundles over the target space in a manner co-determined by the structure of the 2-category of abelian bundle gerbes with connection over it. The construct admits a neat interpretation in terms of a relative Cartan calculus associated with the hierarchy of manifolds that compose the target space of the multi-phase sigma model. The paper also discusses at length the gauge anomaly for the rigid symmetries, derived and quantified cohomologically in a previous work of Gawędzki, Waldorf and the author. The ensuing reinterpretation of the small gauge anomaly in terms of the twisted relative Courant algebroid modelling the Poisson algebra of Noether charges of the symmetries is elucidated through an equivalence between a category built from data of the gauged sigma model and that of principal bundles over the world-sheet with a structural action groupoid based on the target space. Finally, the large gauge anomaly is identified with the obstruction to the existence of topological defect networks implementing the action of the gauge group of the gauged sigma model and those giving a local trivialisation of a gauge bundle of an arbitrary topology over the world-sheet.

Contents

Key words and phrases. Sigma models, dualities, defects; Gauge anomaly; Gerbes; Generalised geometry; Algebroids and groupoids; Principal bundles with structural groupoid.

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1. Introduction

Ever since the seminal contributions by Noether and Wigner, precise identification and subsequent investigation of symmetries of the physical system, both in the classical and in the quantum régime, has become physicists' obsession as one of the most fundamental and effective tools of a systematic construction and exploration of mathematical models of physical phenomena. The numerous manifestations of the Symmetry Principle include the structuring of the state space of the physical theory in terms of the representation theory of the relevant current algebra and the constraining of the analytic form of correlation functions of the quantised theory with the help of the Ward–Takahashi identities. Within the framework of local field theory, the Symmetry Principle is invariably accompanied by the Gauge Principle which stipulates that global (or rigid) symmetries of the theory be rendered local, whereupon the theory be descended (or reduced) to the 'physical' space of orbits of the action of the thus engendered gauge group. This gauging procedure can meet with obstructions – the so-called gauge anomalies – whose analysis has served to restrict the range of admissible models of quantum field theory, working as a super-selection rule for interaction schemes consistent with the assumed gauge invariance.

The concept of symmetry develops novel geometric and cohomological aspects in the context of multi-phase non-linear σ-models, with the structure of a metric manifold on the fibre – termed the target space – of the covariant configuration bundle[1] extended, upon incorporation of the so-called (topological) Wess–Zumino interaction term in the action functional, to include a geometric realisation of a distinguished class in an appropriate (relative) real Deligne hypercohomology group of the target space. The coexistence of distinct phases of the field theory is marked by embedding in its space-time codimension-1 loci of field discontinuity – termed domain walls or defects – carrying cohomological data, pulled back from the target space, that ensure invariance of the multi-phase σ-model under those space-time diffeomorphisms which preserve the defect. The presence of a smooth structure on the target space prompts questions as to the existence of a geometric (that is algebroidal resp. groupoidal) target-space model, understood as a pre-image under a structure-preserving map, of the canonical presentation of rigid symmetries of the σ-model on the state space of the latter, be it in their infinitesimal form (through Noether hamiltonians) or in the finite form (through automorphisms of the space of states). The obvious measure of naturalness of such a symmetry model is its compatibility with the hypercohomological structure over the target space necessitated by a rigorous definition of the Wess–Zumino term, as well as a simple interpretation of the gauge anomaly furnished by it. The multi-phase character of the field theories of interest, and – in particular – the defect-duality correspondence established in Ref. [Sus11], impose further coherence constraints on an admissible symmetry model as they suggest the emergence of natural relations (or morphisms, in an appropriate category), of an intrinsically cohomological quality, between symmetry models assigned to the phases of the field theory that are mapped to one another across those special defects – termed symmetric – which are transmissive to the symmetry currents of the respective phases. These relations are – in turn – subject to secondary constraints at defect self-intersections, expressing compatibility of their definition with trans-defect splitting-joining interactions (represented by non-trivial space-time topologies). The said compatibility conditions correspond, in the canonical description, to the requirement that there exist an intertwiner, induced from the data pulled back to the intersections from the target space, between representations of the symmetry algebra resp. group carried by the phases converging at the defect intersection.

A methodical derivation of the target-space symmetry model and verification of its naturalness (in the two-dimensional setting) is the main objective of the present paper. It is attained through elaboration and essential extension, to the multi-phase setting of interest, of the earlier results – obtained by Alekseev and Strobl in Ref. [AS05] – on the (Courant-)algebroidal nature of the model for infinitesimal symmetries of the mono-phase field theory, with the underlying hypercohomological structure encoded by the Hitchin isomorphisms of Ref. [Hit03]. Instrumental in the construction is the canonical description of the multi-phase σ-model set up in the companion work [Sus11]. Further structural background and guiding insights are provided by the works [GSW10, GSW12] of Gawędzki, Waldorf and the author on the geometry and cohomology of the gauge anomaly of the two-dimensional non-linear σ-model with the Wess–Zumino term, in which a proposal was advanced, and subsequently backed up by ample evidence in its favour, for the target-space model of finite symmetries amenable to

[1] Recall that the bundle is defined as the fibre bundle over the spacetime of the field theory under consideration whose sections are precisely the lagrangean fields of the theory.

gauging. It is given by an equivariant structure on the string background of the σ-model, composed of a self-coherent collection of 0-, 1- and 2-cells of the 2-category of abelian bundle gerbes with connection over the nerve of the (symmetry) action groupoid based on the target space of the σ-model. In this language, the gauge anomaly is to be understood as the topological obstruction to the existence of some such equivariant structure. Its careful reappraisal from the vantage point offered by the target-space models for infinitesimal and finite rigid symmetries developed in the present paper, in conjunction with a correspondence (also worked out hereunder) between a category naturally associated with data of the gauged σ-model[2] and the category of principal bundles over the σ-model space-time with a distinguished structural Lie groupoid, demonstrates the necessity of coupling gauge fields of arbitrary topology to the string background of the σ-model and yields a conclusive corroboration of the proposal of Refs. [GSW10, GSW12], mentioned above, for the necessary and sufficient structure with which to endow the string background when gauging its (finite) rigid symmetries. The basic idea employed in the proof of the proposal consists in reinterpreting the gauge anomaly in terms of the obstruction to the existence of a local trivialisation of a gauge bundle of arbitrary topology over the space-time of the multi-phase σ-model. A minor (technical) variation on the same idea permits to approach and understand the gauge anomaly from yet another angle, to wit, as an obstacle to implementing – in the spirit of the defect-duality correspondence – the local (*i.e.* gauged) action of the symmetry group in the gauged σ-model through topological defect networks with data carried by defect junctions of arbitrary valence canonically induced, in the manner discussed in Refs. [RS09] and [Sus11], from those carried by the elementary 3-valent ones. The latter construction is to be seen as an explicit realisation of the concept, put forward in Ref. [Sus11], of a simplicial duality background, consistent with the definition, extracted from the categorial quantisation scheme in Ref. [FFRS09], of the conformal field theory reduced to the orbit space of the action of the symmetry group on the target space of the parent σ-model.

We conclude this section with an outline of the contents of the present paper. Thus, in Section 2, some basic generalised-geometric constructs are introduced that capture the algebra of infinitesimal rigid symmetries of the mono-phase σ-model (in an arbitrary space-time dimension), and the underlying hypercohomological structure is discussed. In Section 3, the formalism from the previous section is specialised to the two-dimensional setting of immediate interest and interpreted as a target-space model of the Poisson (resp. commutator) algebra of Noether hamiltonians of the rigid symmetry on the state space of the mono-phase σ-model. Section 4 examines the issue of identification of the Noether hamiltonians for the two phases of the σ-model set in correspondence by a conformal defect in the algebraic and canonical frameworks set up in the preceding sections. Section 5 gives an extension of the target-space model for infinitesimal symmetries valid in the presence of non-intersecting (symmetric) defect lines. In Section 6, circumstances are examined in which Noether charges of the rigid symmetry are additively conserved in the cross-defect splitting-joining interactions. In Section 7, the construction of the target-space model for (infinitesimal) symmetries of an arbitrary multi-phase σ-model is completed and subsequently reinterpreted in the framework of a relative differential (Cartan) calculus for the hierarchy of smooth manifolds that compose the target space of the σ-model. Section 8 contains a comprehensive analysis of the various canonical and geometric facets of the gauge anomaly, culminating in a hands-on construction of the topological defect network implementing the local action of the symmetry group in the gauged multi-phase σ-model, as well as a local (space-time) description of that σ-model coupled to a gauge field of an arbitrary topology. Finally, Section 9 recapitulates the main result of the paper and lists some outstanding related problems that deserve, in the author's opinion, to be addressed in near future.

The present paper is to be viewed as a direct continuation of the companion work [Sus11] to which it makes frequent reference, borrowing the notation, invoking the definitions, and making explicit use – without additional preparations – of some of the constructions. In view of this intimate relation between the two papers, and for the reader's convenience, detailed references to [Sus11] have been distinguished by attaching the Roman numeral "I" to the relevant reference labels, as in "Section I.2", "Figure I.7", "Theorem I.5.8" and "Eq. (I.4.10)".

[2] An object of the category of interest is a principal bundle over the space-time of the σ-model with the structure group given by the group G_σ under gauging that has the following extra property: the bundle associated to it through the action of G_σ on the target space of the σ-model (whose existence is assumed in the first place) admits a global section, interpreted as a lagrangean field of the gauged σ-model.

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2. A differential-algebraic structure for the σ-model

The canonical description of the two-dimensional σ-model shares many important structural properties with that of a charged point-like particle in the background of an abelian gauge field coupling to the particle's charge, with the free-loop space of the target space of the σ-model replacing the particle's target space, and the transgression bundle induced by the gerbe playing the rôle of the circle gauge bundle of the point-particle model. From this vantage point, it proves instructive to first generalise the field-theoretic and geometric concepts introduced in Section I.2, thereby gaining insights into certain natural algebraic structures associated with σ-models at large and some interesting interrelations between those structures on the target space and – upon transgression – on the state space of the theory.

Definition 2.1. Let $(\mathscr{M},\mathrm{g})$ be a metric manifold, termed – as in Ref. [Sus11] – the **target space**, with a closed $(n+2)$-form $\mathrm{H}_{(n)}$, $n\in\mathbb{N}$ with periods from $2\pi\mathbb{Z}$ and an n**-gerbe** $\mathcal{G}_{(n)}$ of curvature $\mathrm{curv}(\mathcal{G}_{(n)})=\mathrm{H}_{(n)}$ over it, the latter being understood in the sense of Ref. [Cha98], *i.e.* as a differential-geometric structure representing a class in the Deligne hypercohomology group $\mathbb{H}^{n+1}(\mathscr{M},\mathcal{D}(n+1)^\bullet_{\mathscr{M}})$. Thus, for a choice ${}^{\mathscr{M}}\mathcal{O}$ of an open cover of $\mathscr{M}$, $\mathcal{G}_{(n)}$ is defined by its **local presentation** in terms of a Čech–Deligne $(n+1)$-cochain

$$\mathcal{G}_{(n)}\xrightarrow{\text{loc.}}(B_i,A_{ij},\ldots,g_{i_1i_2\ldots i_{n+2}})=:b_{(n)}\in\mathcal{A}^{n+2,n+1}({}^{\mathscr{M}}\mathcal{O})\,,$$

satisfying the cohomological identity

$$^{\mathscr{M}}D_{(n+1)}b_{(n)}=(\mathrm{H}_{(n)}|_{^{\mathscr{M}}\mathcal{O}_i},0,0,\ldots,1)\,,\tag{2.1}$$

and determined up to **gauge transformations**

$$b_{(n)}\mapsto b_{(n)}+{}^{\mathscr{M}}D_{(n)}\pi_{(n)}\,,\qquad \pi_{(n)}:=(\Pi_i,\Delta_{ij},\ldots,\chi_{i_ii_2\ldots i_{n+1}})\in\mathcal{A}^{n+2,n}({}^{\mathscr{M}}\mathcal{O})\,,$$

all in the conventions set up in Definition I.2.2. The triple will be denoted jointly as $(\mathscr{M},\mathrm{g},\mathcal{G}_{(n)})=:\mathcal{M}_{(n)}$ and termed the n**-target**. Furthermore, let (Ω_{n+1},η) be a closed oriented $(n+1)$-dimensional manifold with an intrinsic minkowskian[3] metric $\eta=\mathrm{diag}(-1,+1,+1,\ldots,+1)$, termed the **world-volume** and embedded in $\mathscr{M}$ by a continuously differentiable map $X:\Omega_{n+1}\to\mathscr{M}$, to be called the **embedding field**. The $(n+1)$**-dimensional non-linear σ-model for embedding fields X with n-target $\mathcal{M}_{(n)}$ on world-volume** (Ω_{n+1},η) is a theory of continuously differentiable maps $X:\Omega_{n+1}\to\mathscr{M}$ determined by the principle of least action applied to the action functional

$$S_\sigma^{(n+1)}[X]=-\tfrac{1}{2}\int_{\Omega_{n+1}}\mathrm{g}_X(\mathsf{d}X\overset{\wedge}{,}\star_\eta\mathsf{d}X)+S_{\mathrm{top}}^{(n+1)}[X]\,,\tag{2.2}$$

in which

- $\mathsf{d}X(\sigma)=\partial_aX^\mu\,\mathsf{d}\sigma^a\otimes\partial_\mu|_{X(\sigma)}$, in local coordinates $\{\sigma^a\}^{a\in\overline{1,n+1}}$ on Ω_{n+1} and $\{X^\mu\}^{\mu\in\overline{1,\dim\mathscr{M}}}$ on $\mathscr{M}$, and the target-space metric is assumed to act on the second factor of the tensor product;
- $\star_\eta$ is the Hodge operator on $\Omega^\bullet(\Omega_{n+1})$ determined by η;
- the topological term

$$S_{\mathrm{top}}^{(n+1)}[X]=-\mathsf{i}\,\log\mathrm{Hol}_{\mathcal{G}_{(n)}}(X)$$

is defined by the hypersurface holonomy $\mathrm{Hol}_{\mathcal{G}_{(n)}}(X)$ of the n-gerbe, which is an obvious generalisation of the (2-)surface holonomy of the (1-)gerbe $\mathcal{G}$ from Definition I.2.7 (with defect contributions dropped), *i.e.* as a Cheeger–Simons differential character for $(n+1)$-dimensional hypersurfaces, determined by a trivialisation of the pullback n-gerbe $X^*\mathcal{G}_{(n)}$ as

$$\log\mathrm{Hol}_{\mathcal{G}_{(n)}}(X)=[X^*\mathcal{G}_{(n)}]\in\check{H}^n\left(\Omega_{n+1},\mathrm{U}(1)\right)\cong\mathrm{U}(1)\,.$$

[3]The definition could readily be extended so as to allow for generic intrinsic metrics of a lorentzian signature. As the ensuing structure of a reparametrisation-invariant σ-model is irrelevant to our considerations, we simply assume that the minkowskian gauge for the intrinsic metric has been fixed in the classical theory. *Cf.* the footnote on p. 9 of Ref. [Sus11].

The last property of the topological term immediately leads to

Proposition 2.2. *Let $\mathcal{M}_{(n)} = (\mathscr{M}, \mathrm{g}, \mathcal{G}_{(n)})$ be an n-target with n-gerbe $\mathcal{G}_{(n)}$ of curvature $\mathrm{H}_{(n)} \in Z^{n+2}(\mathscr{M})$, and let $\mathscr{V}$ be a vector field on $\mathscr{M}$ with a (local) flow $\xi_t : \mathscr{M} \to \mathscr{M}$ (assumed to exist). The variation of the action functional $S_\sigma^{(n+1)}[X]$ of Eq. (2.2) along ξ_t is then given by*

$$\tfrac{\mathsf{d}}{\mathsf{d}t}\big|_{t=0} S_\sigma^{(n+1)}[\xi_t \circ X] = -\tfrac{1}{2} \int_{\Omega_{n+1}} (\mathscr{L}_{\mathscr{V}} \mathrm{g})_X (\mathsf{d}X \overset{\wedge}{,} \star_\eta \mathsf{d}X) + \int_{\Omega_{n+1}} X^* (\mathscr{V} \lrcorner \mathrm{H}_{(n)}) , \qquad (2.3)$$

where $\mathscr{L}_{\mathscr{V}}$ is the Lie derivative in the direction of the vector field $\mathscr{V}$.

Proof. Obvious, through inspection. *Cf.* Ref. [RS09, App. A.2]. □

From Eq. (2.3), we can immediately read off internal (*i.e.* rigid) symmetries of the $(n+1)$-dimensional σ-model.

Corollary 2.3. [RS09, App. A2][GSW10, Cor. 2.2] *In the notation of Proposition 2.2, internal symmetries of the $(n+1)$-dimensional non-linear σ-model for embedding fields X with n-target $\mathcal{M}_{(n)}$ on world-volume (Ω_{n+1}, η) correspond to pairs $(\mathscr{V}, \upsilon)$ composed of a vector field $\mathscr{V} \in \Gamma(\mathsf{T}M)$ that is **Killing** for g,*

$$\mathscr{L}_{\mathscr{V}} \mathrm{g} = 0 ,$$

and an n-form $\upsilon \in \Omega^n(M)$ subject to the constraint

$$\mathsf{d}\upsilon + \mathscr{V} \lrcorner \mathrm{H}_{(n)} = 0 .$$

The last observation points towards a distinguished and natural rôle played by the bundle $\mathsf{T}M \oplus \wedge^n \mathsf{T}^* M \to M$ over the fibre of the covariant configuration bundle of the σ-model in the description of (infinitesimal internal) symmetries of the latter. We shall, next, study the relevant geometric and algebraic constructs in some detail with view to elaborating this issue.

Definition 2.4. Let $\mathscr{M}$ be a smooth manifold of dimension $\dim \mathscr{M} \geq n \in \mathbb{N}$, with tangent bundle $\mathsf{T}\mathscr{M} \to \mathscr{M}$ and cotangent bundle $\mathsf{T}^*\mathscr{M} \to \mathscr{M}$. The **generalised tangent bundle of type $(1,n)$ over $\mathscr{M}$** is the Whitney sum

$$\mathsf{E}^{(1,n)}\mathscr{M} := \mathsf{T}\mathscr{M} \oplus \wedge^n \mathsf{T}^*\mathscr{M} \to \mathscr{M} .$$

The vector bundle

$$\mathsf{E}^{(n,1)}\mathscr{M} := \wedge^n \mathsf{T}\mathscr{M} \oplus \mathsf{T}^*\mathscr{M} \to \mathscr{M} , \qquad n \in \mathbb{N}_{>0} ,$$

dual to $\mathsf{E}^{(1,n)}\mathscr{M}$ through the non-degenerate pairing of sections

$$\langle \cdot, \cdot \rangle \ : \ \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big) \times \Gamma\big(\mathsf{E}^{(n,1)}\mathscr{M}\big) \to C^\infty(\mathscr{M}, \mathbb{R}) \ : \ (\mathscr{V} \oplus \nu, \mathscr{W} \oplus \varpi) \mapsto \mathscr{V} \lrcorner \varpi + \mathscr{W} \lrcorner \nu ,$$

will be termed the **generalised cotangent bundle of type $(n,1)$ over $\mathscr{M}$**. In the distinguished case of $n = 0$, we define

$$\mathsf{E}^{(0,1)}\mathscr{M} := (\mathscr{M} \times \mathbb{R}) \oplus \mathsf{T}^*\mathscr{M} \to \mathscr{M} ,$$

which is, again, dual to $\mathsf{E}^{(1,0)}$ by the non-degenerate pairing of sections

$$\langle \cdot, \cdot \rangle \ : \ \Gamma\big(\mathsf{E}^{(1,0)}\mathscr{M}\big) \times \Gamma\big(\mathsf{E}^{(0,1)}\mathscr{M}\big) \to C^\infty(\mathscr{M}, \mathbb{R}) \ : \ (\mathscr{V} \oplus f, g \oplus \varpi) \mapsto \mathscr{V} \lrcorner \varpi + f \cdot g .$$

The space of smooth sections of the generalised tangent bundle of type $(1,n)$ is equipped with a natural antisymmetric bilinear operation

$$[\, \mathscr{V} \oplus \upsilon \, , \, \mathscr{W} \oplus \varpi \,]_{\mathrm{V}} := [\mathscr{V}, \mathscr{W}] \oplus \big(\mathscr{L}_{\mathscr{V}} \varpi - \mathscr{L}_{\mathscr{W}} \upsilon - \tfrac{1}{2} \mathsf{d}(\mathscr{V} \lrcorner \varpi - \mathscr{W} \lrcorner \upsilon)\big) \qquad (2.4)$$

termed the **Vinogradov bracket** and introduced in Refs. [Vin90, VC92]. In the formula, the bracket in the vector-field component of the right-hand side is the standard Lie bracket of vector fields on $\mathscr{M}$, and we have, in particular,

$$[\, \mathscr{U} \oplus f \, , \, \mathscr{V} \oplus g \,]_{\mathrm{V}} = [\mathscr{U}, \mathscr{V}] \oplus (\mathscr{U}(g) - \mathscr{V}(f)) \qquad (2.5)$$

for sections $\mathscr{U} \oplus f, \mathscr{V} \oplus g \in \Gamma(\mathsf{E}^{(1,0)}\mathscr{M})$. The quadruple $(\mathsf{E}^{(1,n)}\mathscr{M}, [\cdot,\cdot]_{\mathrm{V}}, (\cdot,\cdot)_{\lrcorner}, \alpha_{\mathsf{T}\mathscr{M}}) =: \mathfrak{V}^{(n)}\mathscr{M}$, containing, in addition to the previously described elements, also the symmetric **canonical contraction**

$$(\cdot,\cdot)_{\lrcorner} \ : \ \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big) \times \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big) \to \Omega^{n-1}(\mathscr{M}) \ : \ (\mathscr{V} \oplus \upsilon, \mathscr{W} \oplus \varpi) \mapsto \tfrac{1}{2} (\mathscr{V} \lrcorner \varpi + \mathscr{W} \lrcorner \upsilon) , \quad n > 0$$

$$(\cdot,\cdot)_{\lrcorner} \ : \ \Gamma\big(\mathsf{E}^{(1,0)}\mathscr{M}\big) \times \Gamma\big(\mathsf{E}^{(1,0)}\mathscr{M}\big) \to \{0\} \ : \ (\mathscr{V} \oplus f, \mathscr{W} \oplus g) \mapsto 0 ,$$

$$ \tag{2.6} $$

and the **anchor** $\alpha_{\mathsf{T}\mathscr{M}} : \mathsf{E}^{(1,n)}\mathscr{M} \to \mathsf{T}\mathscr{M}$ given by the canonical projection, will be called the **canonical Vinogradov structure on** $\mathsf{E}^{(1,n)}\mathscr{M}$.

We readily establish the important property

Proposition 2.5. *In the notation of Definition 2.4, let* (f,F) *be an automorphism of* $\mathsf{E}^{(1,n)}\mathscr{M}$ *composed of a diffeomorphism* $f \in \mathrm{Diff}(\mathscr{M})$ *and a (fibre-wise) linear map* $F : \mathsf{E}^{(1,n)}\mathscr{M} \to \mathsf{E}^{(1,n)}\mathscr{M}$ *covering* f *in the sense expressed by the commutative diagram*

$$\begin{tikzcd} \mathsf{E}^{(1,n)}\mathscr{M} \arrow[r, "F"] \arrow[d, "\pi_{\mathsf{E}^{(1,n)}\mathscr{M}}"'] & \mathsf{E}^{(1,n)}\mathscr{M} \arrow[d, "\pi_{\mathsf{E}^{(1,n)}\mathscr{M}}"] \\ \mathscr{M} \arrow[r, "f"] & \mathscr{M} \end{tikzcd} .$$

Suppose also that F *is an automorphism of the Vinogradov structure* $\mathfrak{V}^{(n)}\mathscr{M}$*, i.e.*[4]

$$[\,\cdot\,,\,\cdot\,]_{\mathrm{V}} \circ (F,F) \;=\; F \circ [\,\cdot\,,\,\cdot\,]_{\mathrm{V}}\,, \tag{2.7}$$

$$(\,\cdot\,,\,\cdot\,)_{\lrcorner} \circ (F,F) \;=\; (f^{-1})^* \circ (\,\cdot\,,\,\cdot\,)_{\lrcorner}\,. \tag{2.8}$$

Then, the condition

$$\alpha_{\mathsf{T}\mathscr{M}} \circ F = f_* \circ \alpha_{\mathsf{T}\mathscr{M}}$$

follows automatically, and F *is necessarily of the form*

$$F = \widehat{f} \circ \mathrm{e}^{\mathrm{B}} \circ \widehat{c}_n\,,$$

with

$$\widehat{f} := \begin{pmatrix} f_* & 0 \\ 0 & (f^{-1})^* \end{pmatrix}$$

acting on sections $\mathscr{V} \oplus \upsilon \in \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big)$ *as*

$$\widehat{f} \triangleright (\mathscr{V} \oplus \upsilon) := f_*\mathscr{V} \oplus (f^{-1})^*\upsilon\,,$$

with

$$\mathrm{e}^{\mathrm{B}} := \begin{pmatrix} \mathrm{id}_{\Gamma(\mathsf{T}\mathscr{M})} & 0 \\ \mathrm{B} & \mathrm{id}_{\Omega^n(\mathscr{M})} \end{pmatrix}, \qquad \mathrm{B} \in Z^{n+1}(\mathscr{M}) \tag{2.9}$$

acting as

$$\mathrm{e}^{\mathrm{B}} \triangleright (\mathscr{V} \oplus \upsilon) := \mathscr{V} \oplus (\upsilon + \mathscr{V} \lrcorner \mathrm{B})\,,$$

and with

$$\widehat{c}_n = \begin{pmatrix} \mathrm{id}_{\Gamma(\mathsf{T}\mathscr{M})} & 0 \\ 0 & c^{\delta_{n,0}}\,\mathrm{id}_{\Omega^n(\mathscr{M})} \end{pmatrix}, \qquad c \in \mathbb{R}^\times$$

acting as

$$\widehat{c}_n \triangleright (\mathscr{V} \oplus \upsilon) := \mathscr{V} \oplus c^{\delta_{n,0}} \cdot \upsilon\,.$$

Proof. First of all, note that $\widehat{f}$, in which f_* is the covering map for f on the total space $\mathsf{T}\mathscr{M}$ (and its tensor powers) and $(f^{-1})^*$ has the same interpretation for $\mathsf{T}^*\mathscr{M}$ (whence also its appearance on the right-hand side of Eq. (2.8)), is an automorphism of $\mathfrak{V}^{(n)}\mathscr{M}$. This follows immediately from the identities

$$[\cdot,\cdot] \circ (f_*, f_*) = f_* \circ [\cdot,\cdot]\,,$$

[4] By a slight abuse of the notation, we denote the map on sections by the same symbol as the one used for the bundle map.

written in terms of the Lie bracket $[\cdot,\cdot]$ of vector fields, and from

$$f_*\mathscr{V} \lrcorner (f^{-1})^* \upsilon = (f^{-1})^* (\mathscr{V} \lrcorner \upsilon)\,,$$

the latter being satisfied for arbitrary $\mathscr{V} \in \Gamma(\mathsf{T}\mathscr{M})$ and $\upsilon \in \Omega^n(\mathscr{M})$.

We may, next, consider the automorphism $(\mathrm{id}_{\mathscr{M}}, G) := \widehat{f}^{-1} \circ F$ of $\mathfrak{V}^{(n)}\mathscr{M}$, covering the identity diffeomorphism on $\mathscr{M}$. Let us begin with the case of $n > 0$. Take an arbitrary $g \in C^\infty(\mathscr{M},\mathbb{R})$ and compute, for any $\mathfrak{V},\mathfrak{W} \in \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big)$, the expression

$$[\, g\cdot\mathfrak{V}\,,\,\mathfrak{W}\,]_{\mathrm{V}} = g\cdot[\,\mathfrak{V}\,,\,\mathfrak{W}\,]_{\mathrm{V}} - \alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{W})(g)\cdot\mathfrak{V} + 0 \oplus \mathsf{d}g \wedge (\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner}\,.$$

The assumption that G is an automorphism of $\mathfrak{V}^{(n)}\mathscr{M}$ covering the identity diffeomorphism gives

$$\big[\alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{W})(g) - \alpha_{\mathsf{T}\mathscr{M}}\big(G(\mathfrak{W})\big)(g)\big]\cdot G(\mathfrak{V}) = G\big(0\oplus \mathsf{d}g\wedge(\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner}\big) - 0\oplus\mathsf{d}g\wedge(\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner}\,. \tag{2.10}$$

Upon choosing $\mathfrak{V} = \mathscr{V}\oplus 0$ and $\mathfrak{W} = \mathscr{W}\oplus 0$ for arbitrary vector fields $\mathscr{V},\mathscr{W}$, so that $(\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner} = 0$, the above reduces to

$$\big[\mathscr{W}(g) - \alpha_{\mathsf{T}\mathscr{M}}\big(G(\mathscr{W}\oplus 0)\big)(g)\big]\cdot G(\mathscr{V}\oplus 0) = 0\,. \tag{2.11}$$

Clearly, $G|_{\Gamma(\mathsf{T}\mathscr{M})\oplus\{0\}} \not\equiv 0$ (as an automorphism). Using this, in conjunction with the arbitrariness of g in Eq. (2.11), we conclude that the identity

$$\alpha_{\mathsf{T}\mathscr{M}}\big(G(\mathscr{W}\oplus 0)\big) = \mathscr{W}$$

has to hold true for all $\mathscr{W}\in\Gamma(\mathsf{T}\mathscr{M})$, whence

$$G = \begin{pmatrix} \mathrm{id}_{\Gamma(\mathsf{T}\mathscr{M})} & G_{1,2} \\ G_{2,1} & G_{2,2} \end{pmatrix}$$

for some linear operators

$$G_{1,2}\ :\ \Omega^n(\mathscr{M}) \to \Gamma(\mathsf{T}\mathscr{M})\,,\qquad G_{2,1}\ :\ \Gamma(\mathsf{T}\mathscr{M}) \to \Omega^n(\mathscr{M})\,,$$

$$G_{2,2}\ :\ \Omega^n(\mathscr{M}) \to \Omega^n(\mathscr{M})\,.$$

The second of the three, $G_{2,1}$, is a section of $\mathsf{T}^*\mathscr{M}\otimes\wedge^n\mathsf{T}^*\mathscr{M}$ which is readily seen, via

$$0 \equiv (\,\mathscr{V}\oplus 0\,,\,\mathscr{W}\oplus 0\,)_{\lrcorner} = (\,G(\mathscr{V}\oplus 0)\,,\,G(\mathscr{W}\oplus 0)\,)_{\lrcorner} = \tfrac{1}{2}\,\big(G_{2,1}(\mathscr{V},\mathscr{W},\ldots) + G_{2,1}(\mathscr{W},\mathscr{V},\ldots)\big)\,,$$

valid for arbitrary vector fields $\mathscr{V},\mathscr{W}$, to be an $(n+1)$-form,

$$G_{2,1} =: \mathrm{B} \in \Omega^{n+1}(\mathscr{M})\,.$$

Having established that, take $\mathfrak{V} = \mathscr{V}\oplus\upsilon$ arbitrary and set $\mathfrak{W} = (-\mathscr{V})\oplus\upsilon$, so that, again, $(\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner} = 0$ and Eq. (2.10) yields

$$G_{1,2}(\upsilon)(g)\cdot\big[\big(\mathscr{V} + G_{1,2}(\upsilon)\big)\oplus\big(\mathscr{V}\lrcorner\mathrm{B} + G_{2,2}(\upsilon)\big)\big] = 0\,.$$

The vanishing of the vector-field component on the left-hand side of the above identity implies – in virtue of the arbitrariness of $\mathscr{V}$ and g –

$$G_{1,2}(\upsilon) \equiv 0\,.$$

This ensures that the identity

$$\alpha_{\mathsf{T}\mathscr{M}}\circ G = \alpha_{\mathsf{T}\mathscr{M}}$$

obtains, and so we can rewrite Eq. (2.10) as

$$\mathsf{d}g\wedge(\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner} = G_{2,2}\big(\mathsf{d}g\wedge(\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner}\big)\,.$$

We conclude that

$$G_{2,2} = \mathrm{id}_{\Omega^n(\mathscr{M})}\,.$$

At this stage, it remains to check Eqs. (2.7) and (2.8) for the operator

$$G = \begin{pmatrix} \mathrm{id}_{\Gamma(\mathsf{T}\mathscr{M})} & 0 \\ \mathrm{B} & \mathrm{id}_{\Omega^n(\mathscr{M})} \end{pmatrix}$$

derived above. We find, for arbitrary sections $\mathfrak{V} = \mathscr{V}\oplus\upsilon$ and $\mathfrak{W} = \mathscr{W}\oplus\varpi$ of $\mathsf{E}^{(1,n)}\mathscr{M}$,

$$(\,G(\mathfrak{V})\,,\,G(\mathfrak{W})\,)_{\lrcorner} = \tfrac{1}{2}\,(\mathscr{V}\lrcorner\mathscr{W}\lrcorner\mathrm{B} + \mathscr{V}\lrcorner\varpi + \mathscr{W}\lrcorner\mathscr{V}\lrcorner\mathrm{B} + \mathscr{W}\lrcorner\upsilon) = (\,\mathfrak{V}\,,\,\mathfrak{W}\,)_{\lrcorner}\,,$$

which is the desired result, and

$$\begin{aligned}[]
[\,G(\mathfrak{V})\,,\,G(\mathfrak{W})\,]_{\rm V} &= [\,\mathfrak{V}\,,\,\mathfrak{W}\,]_{\rm V}+0\oplus\big(\mathscr{L}_{\mathscr{V}}(\mathscr{W}\lrcorner{\rm B})-\mathscr{L}_{\mathscr{W}}(\mathscr{V}\lrcorner{\rm B})-\tfrac{1}{2}\,{\sf d}(\mathscr{V}\lrcorner\mathscr{W}\lrcorner{\rm B}-\mathscr{W}\lrcorner\mathscr{V}\lrcorner{\rm B})\big)\\
&= [\,\mathfrak{V}\,,\,\mathfrak{W}\,]_{\rm V}+0\oplus\big([\mathscr{V},\mathscr{W}]\lrcorner{\rm B}-\mathscr{V}\lrcorner\mathscr{W}\lrcorner{\sf d}{\rm B}\big)\equiv G\big([\,\mathfrak{V}\,,\,\mathfrak{W}\,]_{\rm V}\big)-0\oplus\mathscr{V}\lrcorner\mathscr{W}\lrcorner{\sf d}{\rm B}\,,
\end{aligned}$$

from which the thesis of the proposition follows for $n>0$.

Passing to the case of $n=0$, we note that, owing to the triviality of the canonical contraction, Eq. (2.8) is satisfied automatically, and Eq. (2.10) now simplifies as

$$\big[\alpha_{{\sf T}\mathscr{M}}(\mathfrak{W})(g)-\alpha_{{\sf T}\mathscr{M}}\big(G(\mathfrak{W})\big)(g)\big]\cdot G(\mathfrak{V})=0\,. \tag{2.12}$$

Invoking the assumed automorphicity of G, we infer, due to the arbitrariness of $g,\mathfrak{V}$ and $\mathfrak{W}$, that

$$G=\begin{pmatrix} {\rm id}_{\Gamma({\sf T}\mathscr{M})} & 0\\ {\rm B} & C\end{pmatrix}$$

for some ${\rm B}\in\Gamma({\sf T}^*\mathscr{M})$ and $C\in C^\infty(\mathscr{M},\mathbb{R})$. Upon substitution of the above into Eq. (2.7), the latter being evaluated on $\mathfrak{V}=\mathscr{V}\oplus f$ and $\mathfrak{W}=\mathscr{W}\oplus g$, we obtain the condition

$$\mathscr{W}\lrcorner\mathscr{V}\lrcorner{\sf d}{\rm B}+g\cdot\mathscr{V}(C)-f\cdot\mathscr{W}(C)=0\,,$$

which leads to the result

$$G=\begin{pmatrix} {\rm id}_{\Gamma({\sf T}\mathscr{M})} & 0\\ {\rm B} & c\end{pmatrix} \tag{2.13}$$

with ${\rm B}\in Z^1(\mathscr{M})$ and $c\in\mathbb{R}$. The requirement of invertibility of G ultimately fixes the range of c as $\mathbb{R}\setminus\{0\}$ and thereby completes the proof. □

Remark 2.6. In the distinguished case of $n=1$, the generalised tangent bundle of type $(1,n)$ becomes self-dual, the canonical contraction coincides with the duality, and the canonical Vinogradov structure is equivalent to the canonical Courant algebroid of Refs. [Cou90, Dor93, LWX98], with $[\cdot,\cdot]_{\rm V}$ the canonical Courant bracket. Proposition 2.5 then reproduces the classification result of Ref. [Gua03, Prop. 3.24].

In order to put the distinguished case $n=0$ on equal footing with the remaining cases, and – more importantly – with view to subsequent applications of the formalism developed in the context of the two-dimensional σ-model, we specialise the previous definition as

Definition 2.7. In the notation of Definition 2.4 and of Proposition 2.5, a **unital automorphism of generalised tangent bundle** ${\sf E}^{(1,n)}\mathscr{M}$ is an automorphism (f,F) of ${\sf E}^{(1,n)}\mathscr{M}$ with the additional property that

$${\rm pr}_{\Omega^n(\mathscr{M})}\circ F=(f^{-1})^*\circ{\rm pr}_{\Omega^n(\mathscr{M})}\,.$$

Convention 2.8. From now onwards, all morphisms between generalised tangent bundles will be assumed unital. Whenever possible, this will be explicitly marked by a subscript u on the symbols of the relevant morphism sets.

✓

In the presence of an n-gerbe over $\mathscr{M}$, there arises a natural notion of a topological twist of the bundle ${\sf E}^{(1,n)}\mathscr{M}$ and of the algebraic structure $\mathfrak{V}^{(n)}\mathscr{M}$ on it. In general,

Definition 2.9. Adopt the notation of Definition 2.4, and let ${}^{\mathscr{M}}\mathcal{O}=\{{}^{\mathscr{M}}\mathcal{O}_i\}_{i\in\mathscr{I}_{\mathscr{M}}}$ be an open cover of $\mathscr{M}$ with an index set $\mathscr{I}_{\mathscr{M}}$. A **twisted generalised tangent bundle of type** $(1,n)$ **over** $\mathscr{M}$ is a vector bundle ${\sf E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}\to\mathscr{M}$ with a total space locally isomorphic to ${\sf E}^{(1,n)}\mathscr{M}$ and determined by a collection $(\mathfrak{g}_{ij})_{i,j\in\mathscr{I}_{\mathscr{M}}}$ of **transition maps** $\mathfrak{g}_{ij}\in{\rm End}_{\rm u}\big({\sf E}^{(1,n)}\mathscr{M}({}^{\mathscr{M}}\mathcal{O}_{ij})\big)$, ${}^{\mathscr{M}}\mathcal{O}_{ij}={}^{\mathscr{M}}\mathcal{O}_i\cap{}^{\mathscr{M}}\mathcal{O}_j$. The maps are required to cover the identity diffeomorphism on $\mathscr{M}$ and to satisfy the usual cocycle condition

$$(\mathfrak{g}_{jk}\circ\mathfrak{g}_{ij})|_{{}^{\mathscr{M}}\mathcal{O}_{ijk}}=\mathfrak{g}_{ik}|_{{}^{\mathscr{M}}\mathcal{O}_{ijk}} \tag{2.14}$$

on non-empty triple intersections ${}^{\mathscr{M}}\mathcal{O}_{ijk}={}^{\mathscr{M}}\mathcal{O}_i\cap{}^{\mathscr{M}}\mathcal{O}_j\cap{}^{\mathscr{M}}\mathcal{O}_k$. Thus, the bundle has local sections $\mathfrak{V}_i\in{\sf E}^{(1,n)}\mathscr{M}({}^{\mathscr{M}}\mathcal{O}_i)$ related as per

$$\mathfrak{V}_j|_{{}^{\mathscr{M}}\mathcal{O}_{ij}}=\mathfrak{g}_{ij}\vartriangleright\mathfrak{V}_i|_{{}^{\mathscr{M}}\mathcal{O}_{ij}}$$

on non-empty double intersections ${}^{\mathscr{M}}\mathcal{O}_{ij}$. An **isomorphism between a pair** $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}^\alpha_{ij}\}}\mathscr{M},\ \alpha \in \{1,2\}$ is a collection ${}^{\mathscr{M}}\chi = (\mathfrak{h}_i)_{i\in\mathscr{I}_{\mathscr{M}}}$ of local bundle maps $\mathfrak{h}_i \in \mathrm{End}_{\mathrm{u}}\big(\mathsf{E}^{(1,n)}\mathscr{M}({}^{\mathscr{M}}\mathcal{O}_i)\big)$ covering the identity diffeomorphism on $\mathscr{M}$ and such that

$$\mathfrak{g}^2_{ij} = \mathfrak{h}_j \circ \mathfrak{g}^1_{ij} \circ \mathfrak{h}_i^{-1}\,.$$

These induce maps

$$\mathfrak{V}^2_i = \mathfrak{h}_i \triangleright \mathfrak{V}^1_i \tag{2.15}$$

between the respective local sections $\mathfrak{V}^\alpha_i \in \mathsf{E}^{(1,n)}_{\{\mathfrak{g}^\alpha_{ij}\}}\mathscr{M}({}^{\mathscr{M}}\mathcal{O}_i)$.

Remark 2.10. Twisted generalised tangent bundles (of type $(1,1)$) were first introduced in Ref. [Hit03], *cf.* also Ref. [Bar07], in the restricted form in which the twist was determined by local data of a gerbe, *cf.* Corollary 2.17 for a generalisation of that result.

Remark 2.11. We could also consider more general isomorphisms covering diffeomorphisms between different bases. That, however, while completely straightforward in itself, would necessitate – at least in the present (local) formulation – the introduction of Čech-extended manifold maps (in the sense of Ref. [RS09]), a complication that we choose to avoid here.

We augment the above definition with

Definition 2.12. In the notation of Definitions 2.4 and 2.9, a **local Vinogradov structure on** $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ is a collection of Vinogradov structures $\big(\mathsf{E}^{(1,n)}({}^{\mathscr{M}}\mathcal{O}_i), [\,\cdot\,,\,\cdot\,]_{\mathrm{V}}\,, (\,\cdot\,,\,\cdot\,)_{\lrcorner}\,, \alpha_{\mathsf{T}\mathscr{M}}\big)$ over components ${}^{\mathscr{M}}\mathcal{O}_i$ of ${}^{\mathscr{M}}\mathcal{O}$. We say that there exists a **global Vinogradov structure on** $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ iff the transition maps $\mathfrak{g}_{ij}$ map the local sections *homomorphically* into one another, so that the Vinogradov bracket of local sections $\mathfrak{V}_i, \mathfrak{W}_i \in \mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}({}^{\mathscr{M}}\mathcal{O}_i)$ is also a local section over ${}^{\mathscr{M}}\mathcal{O}_i$,

$$\mathfrak{g}_{ij} \triangleright [\,\mathfrak{V}_i\,,\,\mathfrak{W}_i\,]_{\mathrm{V}} = [\,\mathfrak{V}_j\,,\,\mathfrak{W}_j\,]_{\mathrm{V}}\,.$$

An **isomorphism between global Vinogradov structures** $\mathfrak{V}^{(n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M} := \Big(\mathsf{E}^{(1,n)}_{\{\mathfrak{g}^\alpha_{ij}\}}\mathscr{M}, [\,\cdot\,,\,\cdot\,]^\alpha_{\mathrm{V}}\,, (\,\cdot\,,\,\cdot\,)^\alpha_{\lrcorner}\,, \alpha_{\mathsf{T}\mathscr{M}}\Big)$ is an isomorphism ${}^{\mathscr{M}}\chi\ :\ \mathsf{E}^{(1,n)}_{\{\mathfrak{g}^1_{ij}\}}\mathscr{M} \xrightarrow{\cong} \mathsf{E}^{(1,n)}_{\{\mathfrak{g}^2_{ij}\}}\mathscr{M}$ that lifts to a homomorphism of the respective local Vinogradov structures as per

$$\begin{aligned} [\,\cdot\,,\,\cdot\,]^2_{\mathrm{V}} \circ (\mathfrak{h}_i, \mathfrak{h}_i) &= \mathfrak{h}_i \circ [\,\cdot\,,\,\cdot\,]^1_{\mathrm{V}}\,, \\ (\,\cdot\,,\,\cdot\,)^2_{\lrcorner} \circ (\mathfrak{h}_i, \mathfrak{h}_i) &= (\,\cdot\,,\,\cdot\,)^1_{\lrcorner}\,. \end{aligned}$$

We readily establish

Proposition 2.13. *In the notation of Definitions 2.9 and 2.12, with* ${}^{\mathscr{M}}\mathcal{O}$ *a* good *open cover of* $\mathscr{M}$*, there exists a global Vinogradov structure* $\mathfrak{V}^{(n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ *on* $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M} \to \mathscr{M}$ *iff the transition maps of the bundle can be written as*

$$\mathfrak{g}_{ij} := \mathrm{e}^{(-1)^n\,\mathsf{d}A_{ij}} \tag{2.16}$$

for some $A_{ij} \in \Omega^n({}^{\mathscr{M}}\mathcal{O}_{ij})$ *such that*

$$(A_{jk} - A_{ik} + A_{ij})|_{{}^{\mathscr{M}}\mathcal{O}_{ijk}} = (-1)^n\,\mathsf{d}^{(n)}C_{ijk} \tag{2.17}$$

for some $C_{ijk} \in \Omega^{n-1}({}^{\mathscr{M}}\mathcal{O}_{ijk})$*. Here,* $\mathsf{d}^{(n)} = \mathsf{d}$ *for all* $n \neq 0$*, and* $\mathsf{d}^{(0)}$ *is the trivial embedding of the sheaf of locally constant* $\mathbb{R}$*-valued functions into the sheaf of locally smooth* $\mathbb{R}$*-valued functions, cf. Section 7.1. There exists an isomorphism between a pair* $\mathfrak{V}^{(n)}_{\{\mathfrak{g}^\alpha_{ij}\}}\mathscr{M},\ \alpha \in \{1,2\}$ *of global Vinogradov structures iff there is an isomorphism*

$${}^{\mathscr{M}}\chi = (\mathfrak{h}_i)_{i\in\mathscr{I}_{\mathscr{M}}}\ :\ \mathsf{E}^{(1,n)}_{\{\mathfrak{g}^1_{ij}\}}\mathscr{M} \xrightarrow{\cong} \mathsf{E}^{(1,n)}_{\{\mathfrak{g}^2_{ij}\}}\mathscr{M}\,,$$

understood as in Definition 2.9, with local data of the form

$$\mathfrak{h}_i = \mathrm{e}^{-\mathsf{d}\Pi_i}\,, \tag{2.18}$$

for some $\Pi_i \in \Omega^n({}^{\mathscr{M}}\mathcal{O}_i)$*.*

Proof. A simple consequence of the assumed goodness of the open cover and of Proposition 2.5. □

We can also twist the Vinogradov bracket itself, to wit,

Definition 2.14. Assume the notation of Definition 2.4. The $\Omega_{(n+2)}$**-twisted Vinogradov structure on** $\mathsf{E}^{(1,n)}\mathscr{M}$ is the quadruple $\big(\mathsf{E}^{(1,n)}\mathscr{M}, [\cdot,\cdot]_{\mathrm{V}}^{\Omega_{(n+2)}}, (\cdot,\cdot)_{\lrcorner}, \alpha_{\mathsf{T}\mathscr{M}}\big) =: \mathfrak{V}^{(n),\Omega_{(n+2)}}\mathscr{M}$ in which the antisymmetric bilinear operation $[\cdot,\cdot]_{\mathrm{V}}^{\Omega_{(n+2)}}$ on sections of $\mathsf{E}^{(1,n)}\mathscr{M}$, to be termed the $\Omega_{(n+2)}$**-twisted Vinogradov bracket**, is given by the formula

$$[\mathfrak{V},\mathfrak{W}]_{\mathrm{V}}^{\Omega_{(n+2)}} := [\mathfrak{V},\mathfrak{W}]_{\mathrm{V}} + 0 \oplus \alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{V}) \lrcorner \alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{W}) \lrcorner \Omega_{(n+2)}\,,$$

valid for all $\mathfrak{V},\mathfrak{W} \in \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big)$, and in which all the remaining components are the same as those of the canonical Vinogradov structure $\mathfrak{V}^{(n)}\mathscr{M}$.

An **isomorphism between a pair** $\mathfrak{V}^{(n),\Omega^{\alpha}_{(n+2)}}\mathscr{M}_\alpha,\ \alpha \in \{1,2\}$ is a vector-bundle isomorphism $\chi_{1,2} : \mathsf{E}^{(1,n)}\mathscr{M}_1 \xrightarrow{\cong} \mathsf{E}^{(1,n)}\mathscr{M}_2$ covering a diffeomorphism $h_{1,2} : \mathscr{M}_1 \to \mathscr{M}_2$ that satisfies the identities

$$\begin{aligned}
[\cdot,\cdot]_{\mathrm{V}}^{\Omega^2_{(n+2)}} \circ (\chi_{1,2},\chi_{1,2}) &= \chi_{1,2} \circ [\cdot,\cdot]_{\mathrm{V}}^{\Omega^1_{(n+2)}}\,,\\
(\cdot,\cdot)_{\lrcorner} \circ (\chi_{1,2},\chi_{1,2}) &= (h_{1,2}^{-1})^* \circ (\cdot,\cdot)_{\lrcorner}\,,\\
\alpha_{\mathsf{T}\mathscr{M}_2} \circ \chi_{1,2} &= h_{1,2\,*} \circ \alpha_{\mathsf{T}\mathscr{M}_1}\,.
\end{aligned}$$

Remark 2.15. On specialisation to $n=1$, the last definition reproduces the Courant bracket on the canonical generalised tangent bundle twisted by a 3-form, as introduced in Ref. [ŠW01].

The two twisted structures are related by the following

Proposition 2.16. *In the notation of Definitions 2.4, 2.9 and 2.12, and assuming that* $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ *carries a global Vinogradov structure* $\mathfrak{V}^{(n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$*, the former admits a global trivialisation with local data*

$$\mathfrak{h}_i = \mathrm{e}^{B_i}\,, \qquad B_i \in \Omega^{n+1}({}^{\mathscr{M}}\mathcal{O}_i) \tag{2.19}$$

iff there exists a homomorphism ${}^{\mathscr{M}}\chi = (\mathfrak{h}_i)_{i\in\mathscr{I}_{\mathscr{M}}} : \mathfrak{V}^{(n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M} \to \mathfrak{V}^{(n),\Omega_{(n+2)}}\mathscr{M}$ *between* $\mathfrak{V}^{(n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ *and the* $\Omega_{(n+2)}$*-twisted Vinogradov structure* $\mathfrak{V}^{(n),\Omega_{(n+2)}}\mathscr{M}$ *on* $\mathsf{E}^{(1,n)}\mathscr{M}$ *with the twist given by the global closed* $(n+2)$*-form with restrictions*

$$\Omega_{(n+2)}|_{{}^{\mathscr{M}}\mathcal{O}_i} := \mathsf{d}B_i\,, \tag{2.20}$$

i.e. iff for any two local sections $\mathfrak{V}_i,\mathfrak{W}_i \in \mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}({}^{\mathscr{M}}\mathcal{O}_i)$ *and the corresponding sections* $\mathfrak{V}|_{{}^{\mathscr{M}}\mathcal{O}_i} = \mathfrak{h}_i \triangleright \mathfrak{V}_i$ *and* $\mathfrak{W}|_{{}^{\mathscr{M}}\mathcal{O}_i} = \mathfrak{h}_i \triangleright \mathfrak{W}_i$ *from* $\mathsf{E}^{(1,n)}\mathscr{M}({}^{\mathscr{M}}\mathcal{O}_i)$*, we obtain*

$$[\mathfrak{V},\mathfrak{W}]_{\mathrm{V}}^{\Omega_{(n+2)}} = \mathfrak{h}_i \triangleright [\mathfrak{V}_i,\mathfrak{W}_i]_{\mathrm{V}}\,, \tag{2.21}$$

$$(\mathfrak{V},\mathfrak{W})_{\lrcorner} = (\mathfrak{V}_i,\mathfrak{W}_i)_{\lrcorner} \tag{2.22}$$

$$\alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{V}) = \alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{V}_i)\,. \tag{2.23}$$

Proof. $\Rightarrow$ The bundle $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M} \to \mathscr{M}$ admits a global Vinogradov structure, and so – in virtue of Proposition 2.13 – its transition maps have the form (2.16). Their trivialisation in terms of the $\mathfrak{h}_i$ given in Eq. (2.19) yields the equalities

$$\mathsf{d}A_{ij} = (-1)^{n+1}(B_j - B_i)|_{{}^{\mathscr{M}}\mathcal{O}_{ij}}\,,$$

and so, in particular, the $\mathsf{d}B_i$ define a global $(n+2)$-form $\Omega_{(n+2)}$ on $\mathscr{M}$ as per Eq. (2.20). Using the results from the proof of Proposition 2.5, the trivialisation ${}^{\mathscr{M}}\chi = (\mathfrak{h}_i)_{i\in\mathscr{I}_{\mathscr{M}}}$ is readily checked to define the desired homomorphism (note that the twist in the definition of $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ is restricted to the component $\wedge^n\mathsf{T}^*\mathscr{M}$).

$\Leftarrow$ Adducing the same arguments as in the proof of Proposition 2.5 (this time for the Vinogradov brackets $[\cdot,\cdot]_{\rm V}$ and $[\cdot,\cdot]_{\rm V}^{\Omega_{(n+2)}}$), we readily establish that the (unital) homomorphism ${}^{\mathscr{M}}\chi$, whose existence is assumed, is necessarily of the form

$$
{}^{\mathscr{M}}\chi\vert_{\mathscr{M}\mathcal{O}_i}={\rm e}^{B_i}\,.
$$

It is then clear that the local automorphisms $\mathfrak{h}_i:={}^{\mathscr{M}}\chi\vert_{\mathscr{M}\mathcal{O}_i}$ of $\mathsf{E}^{(1,n)}\mathscr{M}$ determine a trivialisation of $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ via

$$
\mathfrak{g}_{ij}=\bigl(\mathfrak{h}_j^{-1}\circ\mathfrak{h}_i\bigr)\vert_{\mathscr{M}\mathcal{O}_{ij}}\,.
$$

□

We then immediately establish

Corollary 2.17. *In the notation of Definitions 2.1, 2.4 and 2.9, the n-gerbe $\mathcal{G}_{(n)}$ canonically defines a twisted generalised tangent bundle $\mathsf{E}^{(1,n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}$ over $\mathscr{M}$ with a global Vinogradov structure, via*

$$
\mathfrak{g}_{ij}={\rm e}^{(-1)^n\,\mathsf{d}A_{ij}}\,.
$$

The latter structure is homomorphic to the ${\rm H}_{(n)}$-twisted Vinogradov structure on $\mathsf{E}^{(1,n)}\mathscr{M}$ as per

$$
{}^{\mathscr{M}}\chi\ :\ \mathfrak{V}^{(n)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}\to\mathfrak{V}^{(n),\Omega_{(n+2)}}\mathscr{M}\,,\qquad\qquad{}^{\mathscr{M}}\chi\vert_{\mathscr{M}\mathcal{O}_i}={\rm e}^{B_i}\,.
$$

*A (trivially) twisted generalised tangent bundle of the type described will be denoted as $\mathsf{E}^{(1,n)}_{\mathcal{G}_{(n)}}\mathscr{M}$ and termed the $\mathcal{G}_{(n)}$**-twisted generalised tangent bundle of type** $(1,n)$ **over** $\mathscr{M}$. Analogously, the corresponding global Vinogradov structure will be denoted by $\mathfrak{V}^{(n)}_{\mathcal{G}_{(n)}}\mathscr{M}$.*

Remark 2.18. The statement of the corollary clearly makes sense as the transition maps satisfy the standard cocycle condition on triple intersections ${}^{\mathscr{M}}\mathcal{O}_{ijk}$ in consequence of Eq. (2.1),

$$
(\mathfrak{g}_{jk}\circ\mathfrak{g}_{ij})\vert_{\mathscr{M}\mathcal{O}_{ijk}}={\rm e}^{(-1)^n\,\mathsf{d}(A_{ij}+A_{jk})\vert_{\mathscr{M}\mathcal{O}_{ijk}}}={\rm e}^{(-1)^n\,\mathsf{d}A_{ik}\vert_{\mathscr{M}\mathcal{O}_{ijk}}}=\mathfrak{g}_{ik}\vert_{\mathscr{M}\mathcal{O}_{ijk}}\,,
$$

and gauge-equivalent choices of a local presentation of $\mathcal{G}_{(n)}$, as described in Definition 2.1, yield isomorphic bundles,

$$
b_{(n)}\mapsto b_{(n)}+D_{(n)}\pi_{(n)}\qquad\Longrightarrow\qquad(\mathfrak{g}_{ij},\mathfrak{V}_i)\mapsto\bigl(\mathfrak{h}_j\circ\mathfrak{g}_{ij}\circ\mathfrak{h}_i^{-1},\mathfrak{h}_i\triangleright\mathfrak{V}_i\bigr)\,,
$$

with $\mathfrak{h}_i$ as in Eq. (2.18).

The structures introduced in the foregoing paragraphs have an immediate physical realisation, which we state as

Proposition 2.19. *In the notation of Definitions 2.1, 2.4 and 2.14,*

i) *internal symmetries of the $(n+1)$-dimensional σ-model of Definition 2.1 correspond to those smooth sections $\mathfrak{V}$ of $\mathsf{E}^{(1,n)}\mathscr{M}$ which are **Killing** for ${\rm g}$,*

$$
\mathscr{L}_{\alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{V})}{\rm g}=0\,,
$$

and belong to the kernel of the linear differential operator

$$
\mathsf{d}_{{\rm H}_{(n)}}\ :\ \Gamma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)\to\Omega^{n+1}(\mathscr{M})\ :\ \mathscr{V}\oplus\upsilon\mapsto\mathsf{d}\upsilon+\mathscr{V}\lrcorner\,{\rm H}_{(n)}\,;
$$

*we shall call these sections σ**-symmetric** and denote the corresponding subset in $\Gamma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)$ as $\Gamma_\sigma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)$;*

ii) *the ${\rm H}_{(n)}$-twisted Vinogradov bracket $[\cdot,\cdot]_{\rm V}^{{\rm H}_{(n)}}$ closes on $\Gamma_\sigma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)$,*

$$
\mathfrak{V},\mathfrak{W}\in\Gamma_\sigma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)\qquad\Longrightarrow\qquad[\mathfrak{V},\mathfrak{W}]_{\rm V}^{{\rm H}_{(n)}}\in\Gamma_\sigma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)\,,
$$

and every other bracket $[\![\cdot,\cdot]\!]_\sigma$ on $\Gamma_\sigma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)$ with this property and such that

$$
\alpha_{\mathsf{T}\mathscr{M}}\circ[\![\cdot,\cdot]\!]_\sigma=[\cdot,\cdot]\circ(\alpha_{\mathsf{T}\mathscr{M}},\alpha_{\mathsf{T}\mathscr{M}})\tag{2.24}
$$

differs from $[\cdot,\cdot]_{\rm V}^{{\rm H}_{(n)}}$ by a linear operator $\Delta:\Gamma_\sigma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)\wedge\Gamma_\sigma\bigl(\mathsf{E}^{(1,n)}\mathscr{M}\bigr)\to Z^n(\mathscr{M})$.

Proof. Ad i) This is a corollary to Proposition 2.2. Note that the n-form component of a σ-symmetric section is determined up to a closed n-form.

Ad ii) First of all, note that $\alpha_{\mathsf{T}\mathscr{M}}(\ker \mathsf{d}_{\mathrm{H}_{(n)}})$ is a Lie subalgebra of the Lie algebra of vector fields on $\mathscr{M}$ as

$$\mathscr{V}_\alpha \lrcorner \mathrm{H}_{(n)} = -\mathsf{d}\upsilon_\alpha\,, \quad \alpha \in \{1,2\} \qquad \Longrightarrow \qquad [\mathscr{V}_1, \mathscr{V}_2] \lrcorner \mathrm{H}_{(n)} = \mathscr{L}_{\mathscr{V}_1}(\mathscr{V}_2 \lrcorner \mathrm{H}_{(n)}) = -\mathsf{d}(\mathscr{L}_{\mathscr{V}_1}\upsilon_2)\,.$$

This demonstrates the naturalness of constraints (2.24). Having noted that, take an arbitrary pair $\mathfrak{V} = \mathscr{V} \oplus \upsilon, \mathfrak{W} = \mathscr{W} \oplus \varpi \in \Gamma_\sigma(\mathsf{E}^{(1,n)}\mathscr{M})$, so that

$$\mathscr{V} \lrcorner \mathrm{H}_{(n)} = -\mathsf{d}\upsilon\,, \qquad \mathscr{W} \lrcorner \mathrm{H}_{(n)} = -\mathsf{d}\varpi\,,$$

and hence also

$$\mathscr{L}_{\mathscr{V}}\mathrm{H}_{(n)} = 0 = \mathscr{L}_{\mathscr{W}}\mathrm{H}_{(n)}$$

due to the closedness of $\mathrm{H}_{(n)}$. The exterior derivative of the n-form component of the $\mathrm{H}_{(n)}$-twisted Vinogradov bracket

$$[\mathfrak{V}, \mathfrak{W}]_{\mathrm{V}}^{\mathrm{H}_{(n)}} = [\mathscr{V}, \mathscr{W}] \oplus \big(\mathscr{L}_{\mathscr{V}}\varpi - \mathscr{L}_{\mathscr{W}}\upsilon - \tfrac{1}{2}\,\mathsf{d}(\mathscr{V} \lrcorner \varpi - \mathscr{W} \lrcorner \upsilon) + \mathscr{V} \lrcorner \mathscr{W} \lrcorner \mathrm{H}_{(n)}\big)\,. \tag{2.25}$$

reads

$$\begin{aligned}\mathsf{d}(\mathscr{L}_{\mathscr{V}}\varpi - \mathscr{L}_{\mathscr{W}}\upsilon + \mathscr{V} \lrcorner \mathscr{W} \lrcorner \mathrm{H}_{(n)}) &= \mathscr{L}_{\mathscr{V}}\mathsf{d}\varpi + \mathsf{d}(-\mathscr{W} \lrcorner \mathsf{d}\upsilon + \mathscr{V} \lrcorner \mathscr{W} \lrcorner \mathrm{H}_{(n)}) \\ &= -\mathscr{L}_{\mathscr{V}}(\mathscr{W} \lrcorner \mathrm{H}_{(n)}) = -\alpha_{\mathsf{T}\mathscr{M}}\big([\mathfrak{V}, \mathfrak{W}]_{\mathrm{V}}^{\mathrm{H}_{(n)}}\big) \lrcorner \mathrm{H}_{(n)}\end{aligned}$$

as claimed. Furthermore, by assumption,

$$[\mathfrak{V}, \mathfrak{W}]_{\mathrm{V}}^{\mathrm{H}_{(n)}} - [\![\mathfrak{V}, \mathfrak{W}]\!]_\sigma = 0 \oplus \Delta(\mathfrak{V} \wedge \mathfrak{W})$$

for some $\Delta : \Gamma_\sigma(\mathsf{E}^{(1,n)}\mathscr{M}) \wedge \Gamma_\sigma(\mathsf{E}^{(1,n)}\mathscr{M}) \to \Omega^n(\mathscr{M})$, and the previous result implies

$$\mathsf{d}\Delta(\mathfrak{V} \wedge \mathfrak{W}) = \big(\alpha_{\mathsf{T}\mathscr{M}}([\![\mathfrak{V}, \mathfrak{W}]\!]_\sigma) - \alpha_{\mathsf{T}\mathscr{M}}([\mathfrak{V}, \mathfrak{W}]_{\mathrm{V}})\big) \lrcorner \mathrm{H}_{(n)} = 0\,,$$

thereby completing the proof of statement ii).

□

Remark 2.20. It is completely straightforward, at least on the formal level, to pass to the canonical or even pre-quantum description of the $(n+1)$-dimensional non-linear σ-model, in which the n-gerbe $\mathcal{G}_{(n)}$ plays a rôle analogous to that of the (1-)gerbe in the familiar two-dimensional case, that is, in particular, it canonically defines – via a higher-dimensional variant of the transgression map – a pre-quantum bundle of the theory. There then ensues a natural transgression scheme between the attendant Vinogradov structures on the target space of the σ-model and on its state space, in which the canonical contraction enters through the definition of Noether currents and the corresponding hamiltonian functions, and which identifies the $\mathrm{H}_{(n)}$-twisted Vinogradov structure on the generalised tangent bundle $\mathsf{E}^{(1,n)}M$ as the sought-after algebraic counterpart of the canonical Vinogradov structure on the state space of the $(n+1)$-dimensional σ-model mentioned in the Introduction. Instead of pursuing this issue at the hitherto level of generality, we specialise our analysis directly to the case of immediate interest, that is to the two-dimensional σ-model, leaving the generalisation as a simple exercise.

3. Symmetries of the two-dimensional σ-model – the untwisted sector

Having extracted the concept of the generalised tangent bundle from the lagrangean analysis of infinitesimal rigid symmetries of the $(n+1)$-dimensional σ-model (with a topological term), we shall next pose the question as to the rôle played by that concept in the canonical treatment of the symmetries, based on an explicit reconstruction of the phase space of the σ-model, understood as a (pre-)symplectic manifold, in the so-called first-order formalism of Refs. [Gaw72, Kij73, Kij74, KS76, Szc76, KT79] reported in Ref. [Sus11]. Our discussion will enable us to regard the structure of a twisted Courant algebroid on the set of σ-symmetric sections as a homomorphic target-space preimage of the structure of a Poisson algebra on the set of the associated Noether hamiltonians on the phase space of the σ-model.

Our passage to the phase space of the σ-model will be seen to serve yet another purpose, to wit, that of demystifying the emergence of the generalised geometry in the field-theoretic setting of interest. The underlying idea is laid out in

Proposition 3.1. *Adopt the notation of Definition 2.4 and let* P *be a smooth manifold endowed with the structure of a symplectic manifold* (P,Ω) *by a closed non-degenerate 2-form* Ω*. To every* ***hamiltonian function*** $h\in C^\infty(\mathsf{P},\mathbb{R})$*, i.e. a smooth function on* P*, there is associated a smooth section*

$$\mathfrak{X}_h=\mathscr{X}_h\oplus h$$

of the generalised tangent bundle $\mathsf{E}^{(1,0)}\mathsf{P}$ *from the kernel of the linear differential operator*

$$\mathsf{d}_\Omega\ :\ \Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}\big)\to\Gamma(\mathsf{T}^*\mathsf{P})\ :\ \mathscr{X}\oplus f\mapsto\mathsf{d}f+\mathscr{X}\lrcorner\,\Omega\,.$$

Elements of $\ker\mathsf{d}_\Omega$ *will be called* ***hamiltonian sections of*** $\mathsf{E}^{(1,0)}\mathsf{P}$*, and a smooth vector field* $\mathscr{X}_h$ *associated to* h *as indicated above is termed a* ***globally hamiltonian vector field****. The linear map*

$$\mathfrak{X}\ :\ C^\infty(\mathsf{P},\mathbb{R})\to\Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}\big)\ :\ h\mapsto\mathfrak{X}_h$$

determines a homomorphism between the Lie algebra $(C^\infty(\mathsf{P},\mathbb{R}),\{\cdot,\cdot\}_\Omega)$ *of hamiltonian functions with the Lie bracket given by the Poisson bracket induced by* Ω*,*

$$\{h_1,h_2\}_\Omega\coloneqq\mathscr{X}_{h_2}\lrcorner\,\mathscr{X}_{h_1}\lrcorner\,\Omega\,,\qquad h_1,h_2\in C^\infty(\mathsf{P},\mathbb{R})\,,$$

and the Lie algebra $\big(\ker\mathsf{d}_\Omega,[\,\cdot\,,\,\cdot\,]^\Omega_{\mathrm{V}}\big)$ *of hamiltonian sections of* $\mathsf{E}^{(1,0)}\mathsf{P}$ *with the Lie bracket given by the* Ω*-twisted Vinogradov bracket*

$$[\,\mathfrak{X}_{h_1}\,,\,\mathfrak{X}_{h_2}\,]^\Omega_{\mathrm{V}}\coloneqq[\mathscr{X}_{h_1},\mathscr{X}_{h_2}]\oplus\big(\mathscr{X}_{h_1}\lrcorner\,\mathsf{d}h_2-\mathscr{X}_{h_2}\lrcorner\,\mathsf{d}h_1+\mathscr{X}_{h_1}\lrcorner\,\mathscr{X}_{h_2}\lrcorner\,\Omega\big)\,.\tag{3.1}$$

The Ω*-twisted Vinogradov bracket is a unique – up to addition of a linear map* $\Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}\big)\wedge\Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}\big)\to\ker\mathsf{d}\subset C^\infty(\mathsf{P},\mathbb{R})$ *to the 0-form component*[5] *– bilinear antisymmetric operation on* $\Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}\big)$ *with the properties*

$$\alpha_{\mathsf{TP}}\circ[\,\cdot\,,\,\cdot\,]^\Omega_{\mathrm{V}}=[\cdot,\cdot]\circ(\alpha_{\mathsf{TP}},\alpha_{\mathsf{TP}})$$

and

$$\mathfrak{X}_1,\mathfrak{X}_2\in\ker\mathsf{d}_\Omega\qquad\Longrightarrow\qquad[\,\mathfrak{X}_1\,,\,\mathfrak{X}_2\,]^\Omega_{\mathrm{V}}\in\ker\mathsf{d}_\Omega\,,$$

written in terms of the anchor $\alpha_{\mathsf{TP}}\ :\ \mathsf{E}^{(1,0)}\mathsf{P}\to\mathsf{TP}$ *(given by the canonical projection). The triple* $(\mathsf{E}^{(1,0)}\mathsf{P},[\,\cdot\,,\,\cdot\,]_{\mathrm{V}}\,,\alpha_{\mathsf{TP}})$ *will be referred to as the* ***canonical Vinogradov struture on*** $\mathsf{E}^{(1,0)}\mathsf{P}$ *henceforth.*

Proof. The statement of the proposition follows directly from the definition of a hamiltonian function, and from the simple property

$$[\mathscr{X}_{h_1},\mathscr{X}_{h_2}]=\mathscr{X}_{\{h_1,h_2\}_\Omega}$$

of hamiltonian vector fields. The Jacobi identity for the Ω-twisted Vinogradov bracket is then a consequence of the same identity for the Poisson bracket. □

As a first step towards our goal, let us specialise our considerations to two dimensions, extending them simultaneously so as to account for the existence of world-sheet defects.

Definition 3.2. Adopt the notation of Definitions I.2.6 and I.2.7., and let $\mathfrak{B}=(\mathcal{M},\mathcal{B},\mathcal{J})$ be a string background with target $\mathcal{M}=(M,\mathrm{g},\mathcal{G})$, $\mathcal{G}$-bi-brane $\mathcal{B}=\big(Q,\iota_\alpha,\omega,\Phi\mid\alpha\in\{1,2\}\big)$ and $(\mathcal{G},\mathcal{B})$-inter-bi-brane $\mathcal{J}=\bigsqcup_{n\in\mathbb{N}_{\geq3}}\big(T_n,\big(\varepsilon_n^{k,k+1},\pi_n^{k,k+1}\mid k\in\overline{1,n}\big),\varphi_n\big)$, supported over target space $\mathscr{F}\coloneqq M\sqcup Q\sqcup T$, $T=\bigsqcup_{n\geq3}T_n$, all as introduced in Definition I.2.1. Moreover, let Γ be a defect quiver from Definition I.2.6. The two-dimensional non-linear σ-model for network-field configurations $(X\,|\,\Gamma)$ in string background $\mathfrak{B}$ on world-sheet (Σ,γ) with defect quiver Γ is a theory of continuously differentiable maps $X\ :\ \Sigma\to\mathscr{F}$ determined by the principle of least action applied to the action functional

$$S_\sigma[(X\,|\,\Gamma);\gamma]\coloneqq-\tfrac{1}{2}\int_\Sigma\mathrm{g}(\mathsf{d}X\overset{\wedge}{,}\star_\gamma\mathsf{d}X)-\mathsf{i}\,\log\mathrm{Hol}_{\mathcal{G},\Phi,(\varphi_n)}(X\,|\,\Gamma)\,.\tag{3.2}$$

[5] Note that the ambiguity in the definition of a bracket with the properties listed is consistent with the ambiguity in the definition of the hamiltonian function.

Proposition 3.3. [Sus11, Props. I.3.11 & I.3.12] *Let* $\mathsf{P}_{\sigma,\varnothing}$ *and* $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ *be the untwisted and 1-twisted state spaces of the two-dimensional non-linear* σ*-model of Definition 3.2, as introduced in Definitions I.3.9 and I.3.10, respectively. A (pre-)symplectic form on* $\mathsf{P}_{\sigma,\varnothing}$ *can be written as*

$$\Omega_{\sigma,\varnothing}[(X,\mathrm{p})] = \int_{\mathbb{S}^1} \mathrm{Vol}\big(\mathbb{S}^1\big) \wedge \big[\delta \mathrm{p}_\mu \wedge \delta X^\mu + 3(X_*\widehat{t})^\lambda\,\mathrm{H}_{\lambda\mu\nu}\delta X^\mu \wedge \delta X^\nu\big] \tag{3.3}$$

in terms of the canonical coordinates (X,p) *on* $\mathsf{P}_{\sigma,\varnothing}$ *and of components* $\mathrm{H}_{\lambda\mu\nu}$ *of the curvature 3-form* H *of the gerbe* $\mathcal{G}$*. Similarly, a (pre-)symplectic form on* $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ *can be written as*

$$\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}[(X,\mathrm{p},q,V)] = \int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big) \wedge \big[\delta \mathrm{p}_\mu \wedge \delta X^\mu + 3(X_*\widehat{t})^\lambda\,\mathrm{H}_{\lambda\mu\nu}\delta X^\mu \wedge \delta X^\nu\big] + \varepsilon\,\omega(q) \tag{3.4}$$

in terms of the canonical coordinates (X,p,q,V) *on* $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ *and of the curvature* ω *of the* $\mathcal{G}$*-bi-brane* $\mathcal{B}$*.*

We are now ready to study the canonical description of internal symmetries of the two-dimensional σ-model.

We start by recapitulating the algebraic structure on the set of sections of the generalised tangent bundle over the target space in the absence of defects. As a specialisation of Corollary 2.17 and Proposition 2.19 to the case $n=1$, we obtain

Corollary 3.4. *Adopt the notation of Definitions 2.4, 2.9, 2.12 and 2.14. Let* $\mathfrak{B}$ *be a string background with target* $\mathcal{M}=(M,\mathrm{g},\mathcal{G})$*, as detailed in Definition I.2.1, and denote by* $\mathrm{H}\in Z^3(M)$ *the curvature of* $\mathcal{G}$*. Infinitesimal rigid symmetries of the two-dimensional non-linear* σ*-model for network-field configurations* $(X\,|\,\varnothing)$ *in string background* $\mathfrak{B}$ *on world-sheet* (Σ,γ) *with an empty defect quiver* $\Gamma=\varnothing$*, as described in Definition I.2.7, correspond to* σ*-symmetric sections* $\mathfrak{V}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)$*,*

$$\mathscr{L}_{\alpha_{\mathsf{T}M}(\mathfrak{V})}\mathrm{g}=0\,,\qquad\qquad \mathsf{d}_{\mathrm{H}}\mathfrak{V}=0\,.$$

The H*-twisted Vinogradov bracket (of the* H*-twisted Vinogradov structure* $\mathfrak{V}^{(1),\mathrm{H}}M$*) closes on* $\Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)$*,*

$$\mathfrak{V},\mathfrak{W}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)\qquad\Longrightarrow\qquad[\,\mathfrak{V}\,,\,\mathfrak{W}\,]^{\mathrm{H}}_{\mathrm{V}}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)\,.$$

Equivalently, given an open cover ${}^M\mathcal{O}=\{\mathcal{O}_i^M\}_{i\in\mathscr{I}_M}$ *of* M *with an index set* $\mathscr{I}_M$*, together with the associated local presentation* $(B_i,A_{ij},g_{ijk})\in\mathcal{A}^{3,2}({}^M\mathcal{O})$ *of* $\mathcal{G}$*, as described in Definition I.2.2, the symmetries can be represented by* σ*-symmetric sections* $(\mathfrak{V}_i)_{i\in\mathscr{I}_M}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\big)$*,*

$$\mathscr{L}_{\alpha_{\mathsf{T}M}(\mathfrak{V}_i)}\mathrm{g}=0\,,\qquad\qquad \mathsf{d}\mathrm{pr}_{\mathsf{T}^*M}(\mathfrak{V}_i)+\mathscr{L}_{\alpha_{\mathsf{T}M}(\mathfrak{V}_i)}B_i=0\,.$$

The Vinogradov bracket (of the global Vinogradov structure $\mathfrak{V}^{(1)}_{\mathcal{G}}M$*, homomorphic to* $\mathfrak{V}^{(1),\mathrm{H}}M$*) closes on* $\Gamma_\sigma\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\big)$*,*

$$(\mathfrak{V}_i)_{i\in\mathscr{I}_M},(\mathfrak{W}_i)_{i\in\mathscr{I}_M}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\big)\qquad\Longrightarrow\qquad\big([\,\mathfrak{V}_i\,,\,\mathfrak{W}_i\,]_{\mathrm{V}}\big)_{i\in\mathscr{I}_M}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\big)\,.$$

It was demonstrated in the proof of statement ii) of Proposition 2.19 that $\alpha_{\mathsf{T}M}(\ker\mathsf{d}_{\mathrm{H}})$ is a Lie subalgebra of the Lie algebra of vector fields, and the same is true for Killing vector fields. Hence, we establish

Proposition 3.5. *Adopt the notation of Definitions 2.4 and 2.14, and of Proposition 2.19. Let* $(\mathscr{M},\mathrm{g})$ *be a metric manifold with a smooth closed 3-form* $\mathrm{H}\in Z^3(\mathscr{M})$*. The subspace* $\alpha_{\mathsf{T}\mathscr{M}}\big(\Gamma_\sigma\big(\mathsf{E}^{(1,1)}\mathscr{M}\big)\big)$ *is a Lie subalgebra, to be denoted as* $\mathfrak{g}_\sigma$*, within the Lie algebra of Killing vector fields on* $(\mathscr{M},\mathrm{g})$*. Fix a basis* $\{\mathscr{K}_A\}_{A\in\overline{1,K_\sigma}}$*,* $K_\sigma=\dim\mathfrak{g}_\sigma$ *in* $\mathfrak{g}_\sigma$ *such that the defining commutation relations*

$$[\mathscr{K}_A,\mathscr{K}_B]=f_{ABC}\,\mathscr{K}_C$$

hold true for some structure constants f_{ABC}*. The corresponding* σ*-symmetric sections*

$$\mathfrak{K}_A=\mathscr{K}_A\oplus\kappa_A\,,\qquad\qquad \mathscr{L}_{\mathscr{K}_A}\mathrm{g}=0\,,\qquad\qquad \mathsf{d}_{\mathrm{H}}\mathfrak{K}_A=0$$

satisfy the relations

$$[\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]^{\mathrm{H}}_{\mathrm{V}}=f_{ABC}\,\mathfrak{K}_C+0\oplus\big(\Delta_{AB}-\mathsf{d}\mathrm{c}_{(AB)}\big)\,,\qquad\qquad \left\{\begin{array}{l}\Delta_{AB}=\mathscr{L}_{\mathscr{K}_A}\kappa_B-f_{ABC}\,\kappa_C\,,\\[6pt] \mathrm{c}_{(AB)}=(\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,)_{\lrcorner}\,.\end{array}\right. \tag{3.5}$$

Proof. Obvious, through inspection. □

Having presented the target-space aspect of internal symmetries of the σ-model action functional in the absence of defects, we may next consider their symplectic realisation on the state space of the untwisted sector of the theory. To this end, we should lift our previous analysis to the symplectic space $(\mathsf{P}_{\sigma,\varnothing} \equiv \mathsf{T}^*\mathsf{L}M, \Omega_{\sigma,\varnothing})$, whereupon it develops in complete analogy to the geometric discussion.

Definition 3.6. Let $\mathscr{M}$ be a smooth manifold, $\mathsf{L}\mathscr{M} = C^\infty(\mathbb{S}^1,\mathscr{M})$ its free-loop space, and

$$\mathrm{ev}_{\mathscr{M}} \;:\; \mathsf{L}\mathscr{M} \times \mathbb{S}^1 \to \mathscr{M}$$

the canonical evaluation map. Given a vector field $\mathscr{V} \in \Gamma(\mathsf{T}\mathscr{M})$, denote by $\xi_t : \mathscr{M} \to \mathscr{M}$ the flow of $\mathscr{V}$. The **loop-space lift of vector field from $\mathscr{M}$** is a linear map

$$\mathsf{L}_* \;:\; \Gamma(\mathsf{T}\mathscr{M}) \to \Gamma(\mathsf{T}\mathsf{L}\mathscr{M}) \;:\; \mathscr{V} \mapsto \mathsf{L}_*\mathscr{V}\,, \qquad\qquad \mathsf{L}^*\mathscr{V}(F)[X] := \tfrac{\mathsf{d}}{\mathsf{d}t}\big|_{t=0} F[\xi_t \circ X]\,,$$

defined for an arbitrary functional F on $\mathsf{L}\mathscr{M}$ and for a free loop X. The **loop-space lift of n-form from $\mathscr{M}$** is the linear map

$$\mathsf{L}^* \quad : \quad \Omega^n(\mathscr{M}) \to \Omega^{n-1}(\mathsf{L}\mathscr{M}) \;:\; \upsilon \mapsto \int_{\mathbb{S}^1} \mathrm{ev}^*_{\mathscr{M}}\,\upsilon\,, \quad n \in \mathbb{N}_{>0}\,,$$

extended to the case $n = 0$ as per

$$\mathsf{L}^* \;:\; C^\infty(\mathscr{M},\mathbb{R}) \to \{0\} \;:\; f \mapsto 0\,. \tag{3.6}$$

Basic properties of the two lifts are summarised in the following

Lemma 3.7. *Adopt the notation of Definition 3.6 and denote by δ the (functional) exterior derivative on $\Omega^\bullet(\mathsf{L}\mathscr{M})$. Then, for arbitrary $\mathscr{V},\mathscr{W} \in \Gamma(\mathsf{T}\mathscr{M})$ and $\upsilon \in \Omega^n(\mathscr{M})$,*

$$\delta\mathsf{L}^*\upsilon = -\mathsf{L}^*\mathsf{d}\upsilon\,, \qquad\qquad \mathsf{L}_*\mathscr{V} \lrcorner \mathsf{L}^*\upsilon = -\mathsf{L}^*(\mathscr{V} \lrcorner \upsilon)\,,$$

$$[\mathsf{L}_*\mathscr{V},\mathsf{L}_*\mathscr{W}] = \mathsf{L}_*[\mathscr{V},\mathscr{W}]\,.$$

Proof. Obvious, through inspection. □

In the next, physically motivated step, we obtain

Lemma 3.8. *Adopt the notation of Definition 3.6 and of Lemma 3.7. The lift L^* induces a lift*

$$\widetilde{\mathsf{L}}^* := \pi^*_{\mathsf{T}^*\mathsf{L}\mathscr{M}} \circ \mathsf{L}^* \;:\; \Omega^n(\mathscr{M}) \to \Omega^{n-1}(\mathsf{L}\mathscr{M})$$

of n-forms on $\mathscr{M}$ to $(n-1)$-forms on the cotangent bundle $\pi_{\mathsf{T}^*\mathsf{L}\mathscr{M}} : \mathsf{T}^*\mathsf{L}\mathscr{M} \to \mathsf{L}\mathscr{M}$. Analogously, L_* induces a canonical lift

$$\widetilde{\mathsf{L}}_* \;:\; \Gamma\big(\mathsf{T}\mathscr{M}\big) \to \Gamma\big(\mathsf{T}(\mathsf{T}^*\mathsf{L}\mathscr{M})\big)$$

of vector fields $\mathscr{V} \in \Gamma\big(\mathsf{T}\mathscr{M}\big)$ to those on $\mathsf{T}^*\mathsf{L}\mathscr{M}$, fixed by the relations

$$\begin{aligned} \pi_{\mathsf{T}^*\mathsf{L}\mathscr{M}\,*}\widetilde{\mathsf{L}}_*\mathscr{V} &= \mathsf{L}_*\mathscr{V}\,, \\ \mathscr{L}_{\widetilde{\mathsf{L}}_*\mathscr{V}}\theta_{\mathsf{T}^*\mathsf{L}\mathscr{M}} &= 0\,, \end{aligned} \tag{3.7}$$

expressed in terms of the canonical 1-form $\theta_{\mathsf{T}^*\mathsf{L}\mathscr{M}}$ on $\mathsf{T}^*\mathsf{L}\mathscr{M}$ given in Eq. (I.3.14). Then, for arbitrary $\mathscr{V},\mathscr{W} \in \Gamma(\mathsf{T}\mathscr{M})$ and $\upsilon \in \Omega^n(\mathscr{M})$,

$$\delta\widetilde{\mathsf{L}}^*\upsilon = -\widetilde{\mathsf{L}}^*\mathsf{d}\upsilon\,, \qquad\qquad \widetilde{\mathsf{L}}_*\mathscr{V} \lrcorner \widetilde{\mathsf{L}}^*\upsilon = -\widetilde{\mathsf{L}}^*(\mathscr{V} \lrcorner \upsilon)\,, \tag{3.8}$$

$$[\widetilde{\mathsf{L}}_*\mathscr{V},\widetilde{\mathsf{L}}_*\mathscr{W}] = \widetilde{\mathsf{L}}_*[\mathscr{V},\mathscr{W}]\,. \tag{3.9}$$

Proof. Obvious, through inspection. □

Remark 3.9. Relation (3.7) ensures that the fibre coordinate p_ν of a point $\psi = (X^\mu,\mathsf{p}_\nu) \in \mathsf{T}^*\mathsf{L}\mathscr{M}$ has the tensorial properties of a component of a 1-form on $\mathscr{M}$. Explicitly, the canonical lift of a vector field $\mathscr{V} = \mathscr{V}^\mu \frac{\partial}{\partial X^\mu} \in \Gamma(\mathsf{T}\mathscr{M})$ can be written in the form

$$\widetilde{\mathsf{L}}_*\mathscr{V}[\psi] = \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1)\,\Big[\mathscr{V}^\mu\big(X(\cdot)\big)\,\tfrac{\delta}{\delta X^\mu(\cdot)} - \mathsf{p}_\mu(\cdot)\,\partial_\nu\mathscr{V}^\mu\big(X(\cdot)\big)\,\tfrac{\delta}{\delta \mathsf{p}_\nu(\cdot)}\Big]\,.$$

The lifts give rise to a simple algebraic structure, namely,

Lemma 3.10. *Adopt the notation of Definitions 2.4, 2.14 and 3.6, and of Lemma 3.8. The pair* $(\mathsf{L}_*, \mathsf{L}^*)$ *induces a linear mapping*

$$\mathsf{L}_{(1,n)} = \begin{pmatrix} \mathsf{L}_* & 0 \\ 0 & \mathsf{L}^* \end{pmatrix} \;:\; \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big) \to \Gamma\big(\mathsf{E}^{(1,n-1)}\mathsf{L}\mathscr{M}\big) \;:\; \mathscr{V} \oplus \upsilon \mapsto \mathsf{L}_*\mathscr{V} \oplus \mathsf{L}^*\upsilon\,, \qquad n \in \mathbb{N}$$

that relates elements of the respective twisted Vinogradov structures $\mathfrak{V}^{(n),\Omega_{n+2}}\mathscr{M}$ *and* $\mathfrak{V}^{(n-1),\mathsf{L}^*\Omega_{n+2}}\mathsf{L}\mathscr{M}$ *as*

$$\begin{aligned} \alpha_{\mathsf{TL}\mathscr{M}} \circ \mathsf{L}_{(1,n)} &= \mathsf{L}_* \circ \alpha_{\mathsf{T}\mathscr{M}}\,, \\ [\cdot,\cdot]_{\mathrm{V}}^{\mathsf{L}^*\Omega_{(n+2)}} \circ \big(\mathsf{L}_{(1,n)}, \mathsf{L}_{(1,n)}\big) &= \mathsf{L}_{(1,n)} \circ [\cdot,\cdot]_{\mathrm{V}}^{\Omega_{(n+2)}}\,, \\ (\cdot,\cdot)_\lrcorner \circ \big(\mathsf{L}_{(1,n)}, \mathsf{L}_{(1,n)}\big) &= -\mathsf{L}^* \circ (\cdot,\cdot)_\lrcorner\,. \end{aligned}$$

The mapping admits an obvious (canonical) extension

$$\widetilde{\mathsf{L}}_{(1,n)} = \begin{pmatrix} \widetilde{\mathsf{L}}_* & 0 \\ 0 & \widetilde{\mathsf{L}}^* \end{pmatrix} \;:\; \Gamma\big(\mathsf{E}^{(1,n)}\mathscr{M}\big) \to \Gamma\big(\mathsf{E}^{(1,n-1)}\mathsf{T}^*\mathsf{L}\mathscr{M}\big)\,, \tag{3.10}$$

that relates elements of the respective twisted Vinogradov structures $\mathfrak{V}^{(n),\Omega_{n+2}}\mathscr{M}$ *and* $\mathfrak{V}^{(n-1),\widetilde{\mathsf{L}}^*\Omega_{(n+2)}}\mathsf{T}^*\mathsf{L}\mathscr{M}$ *as*

$$\alpha_{\mathsf{T}(\mathsf{T}^*\mathsf{L}\mathscr{M})} \circ \widetilde{\mathsf{L}}_{(1,n)} = \mathsf{L}_* \circ \alpha_{\mathsf{T}\mathscr{M}}\,, \tag{3.11}$$

$$[\cdot,\cdot]_{\mathrm{V}}^{\widetilde{\mathsf{L}}^*\Omega_{(n+2)}} \circ \big(\widetilde{\mathsf{L}}_{(1,n)}, \widetilde{\mathsf{L}}_{(1,n)}\big) = \widetilde{\mathsf{L}}_{(1,n)} \circ [\cdot,\cdot]_{\mathrm{V}}^{\Omega_{(n+2)}}\,, \tag{3.12}$$

$$(\cdot,\cdot)_\lrcorner \circ \big(\widetilde{\mathsf{L}}_{(1,n)}, \widetilde{\mathsf{L}}_{(1,n)}\big) = -\widetilde{\mathsf{L}}^* \circ (\cdot,\cdot)_\lrcorner\,. \tag{3.13}$$

Proof. An immediate corollary to Lemmata 3.7 and 3.8. □

Putting together various results obtained so far and those of Ref. [Sus11], we arrive at a conclusion of immediate relevance to the two-dimensional field theory of interest, phrased as

Theorem 3.11. *Adopt the notation of Definition 3.2, of Corollaries 2.17 and 3.4, of Proposition 3.3, and of Lemma 3.8. Let* $\mathcal{L}_{\sigma,\varnothing} \to \mathsf{P}_{\sigma,\varnothing}$ *the pre-quantum bundle of the untwisted sector of the* σ*-model from Corollary I.3.17. The gerbe* $\mathcal{G}$ *canonically induces a linear mapping*

$$\phi_{\sigma,\varnothing} \;:\; \mathsf{E}^{(1,1)}_{\mathcal{G}} M \to \mathsf{E}^{(1,0)}_{\mathcal{L}_{\sigma,\varnothing}} \mathsf{P}_{\sigma,\varnothing}$$

(with the codomain twisted by the 0-gerbe $\mathcal{L}_{\sigma,\varnothing}$*) that relates elements of the respective global Vinogradov structures* $\mathfrak{V}^{(1)}_{\mathcal{G}} M$ *and* $\mathfrak{V}^{(0)}_{\mathcal{L}_{\sigma,\varnothing}} \mathsf{P}_{\sigma,\varnothing}$ *as*

$$\alpha_{\mathsf{TP}_{\sigma,\varnothing}} \circ \phi_{\sigma,\varnothing} = \widetilde{\mathsf{L}}_* \circ \alpha_{\mathsf{T}M}\,, \tag{3.14}$$

$$[\cdot,\cdot]_{\mathrm{V}} \circ \big(\phi_{\sigma,\varnothing}, \phi_{\sigma,\varnothing}\big) = \phi_{\sigma,\varnothing} \circ [\cdot,\cdot]_{\mathrm{V}}\,, \tag{3.15}$$

$$(\cdot,\cdot)_\lrcorner \circ \big(\phi_{\sigma,\varnothing}, \phi_{\sigma,\varnothing}\big) = \widetilde{\mathsf{L}}^* \circ (\cdot,\cdot)_\lrcorner\,. \tag{3.16}$$

Proof. Choose an open cover ${}^M\mathcal{O} = \{\mathcal{O}_i^M\}_{i \in \mathscr{I}_M}$ of M (with an index set $\mathscr{I}_M$), and induce from it an open cover $\mathcal{O}_{\mathsf{T}^*\mathsf{L}M} = \{\mathcal{O}^*_{\mathrm{i}}\}_{\mathrm{i} \in \mathscr{I}_{\mathsf{T}^*\mathsf{L}M}}$ of $\mathsf{T}^*\mathsf{L}M$ in the same manner as in Corollary I.3.17. Fix local presentations: $(B_i, A_{ij}, g_{ijk}) \in \mathcal{A}^{3,2}({}^M\mathcal{O})$ of the gerbe $\mathcal{G}$, and $(\theta_{\sigma,\varnothing\,\mathrm{i}}, \gamma_{\sigma,\varnothing\,\mathrm{ij}}) \in \mathcal{A}^{2,1}(\mathcal{O}_{\mathsf{T}^*\mathsf{L}M})$ of the pre-quantum bundle $\mathcal{L}_{\sigma,\varnothing}$, the latter as in Eq. (I.3.18). Denote by H the curvature of $\mathcal{G}$, and by $\theta_{\mathsf{T}^*\mathsf{L}M}$ the canonical 1-form on $\mathsf{T}^*\mathsf{L}M$ from Eq. (I.3.14). By virtue of Corollary 2.17, there exist homomorphisms of the Vinogradov structures:

$${}^M\chi \;:\; \mathfrak{V}^{(1)}_{\mathcal{G}} M \to \mathfrak{V}^{(1),\mathrm{H}} M\,, \qquad {}^M\chi|_{\mathcal{O}_i^M} = \mathrm{e}^{B_i}$$

and

$${}^{\mathsf{P}_{\sigma,\varnothing}}\chi \;:\; \mathfrak{V}^{(0)}_{\mathcal{L}_{\sigma,\varnothing}} \mathsf{P}_{\sigma,\varnothing} \to \mathfrak{V}^{(0),\Omega_{\sigma,\varnothing}} \mathsf{P}_{\sigma,\varnothing}\,, \qquad {}^{\mathsf{P}_{\sigma,\varnothing}}\chi|_{\mathcal{O}^*_{\mathrm{i}}} = \mathrm{e}^{\theta_{\sigma,\varnothing\,\mathrm{i}}}\,.$$

The linear mapping in question can now be explicitly defined as

$$\phi_{\sigma,\varnothing} := {}^{\mathsf{P}_{\sigma,\varnothing}}\chi^{-1} \circ \mathrm{e}^{\theta_{\mathsf{T}^*\mathsf{L}M}} \circ \widetilde{\mathsf{L}}_{(1,1)} \circ {}^M\chi$$

in terms of the linear mapping $\widetilde{\mathsf{L}}_{(1,1)}$ from Lemma 3.10. The desired algebraic properties of $\phi_{\sigma,\varnothing}$ listed in the proposition can readily be verified by combining the results from Proposition 2.16 and Lemma 3.10. Note, in particular, that the last of them, (3.16), follows from triviality of the canonical contraction on $\mathsf{E}^{(1,0)}_{\mathcal{L}_{\sigma,\varnothing}}\mathsf{P}_{\sigma,\varnothing}$, *cf.* Eq. (3.6). $\square$

The last theorem provides a clear-cut answer to the general question raised in the Introduction to this section as it demonstrates the existence of a straightforward correspondence between the gerbe-induced (Courant-)algebraic structure on the generalised tangent bundle of type $(1,1)$ over the target space of the σ-model and the canonical Vinogradov structure on its untwisted state space. It will be seen to organise the canonical description of internal symmetries of the σ-model under study, to which we turn next.

Proposition 3.12. *Adopt the notation of Definition 2.4, of Corollaries 2.17 and 3.4, of Proposition 2.19, of Theorem 3.11, and of Lemmata 3.7, 3.8 and 3.10. Let $\mathcal{L}_\mathcal{G} \to \mathsf{L}M$ be the transgression bundle of Theorem I.3.16, the latter having local data $(E_\mathsf{i}, G_\mathsf{ij})$, as explicited in the constructive proof of the theorem, written for the open cover $\mathcal{O}_{\mathsf{L}M} = \{\mathcal{O}_\mathsf{i}\}_{\mathsf{i}\in\mathscr{I}_{\mathsf{L}M}}$ of the free-loop space $\mathsf{L}M = C^\infty(\mathbb{S}^1, M)$ from Proposition I.3.13. Write*

$$\mathfrak{T} := 1 \oplus \theta_{\mathsf{T}^*\mathsf{L}M} \in \Gamma\big(\mathsf{E}^{(0,1)}\mathsf{P}_{\sigma,\varnothing}\big)\,,$$

and call the latter object the ***canonical section of*** $\mathsf{E}^{(1,0)}\mathsf{P}_{\sigma,\varnothing}$. *To every smooth σ-symmetric section $\mathfrak{V}$ of $\mathsf{E}^{(1,1)}M$ there is associated a* ***hamiltonian function*** $h_\mathfrak{V}$, *i.e. a smooth function on $\mathsf{P}_{\sigma,\varnothing}$ satisfying the defining relation*

$$\alpha_{\mathsf{TP}_{\sigma,\varnothing}}\big(\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\big) \lrcorner\, \Omega_{\sigma,\varnothing} =: -\delta h_\mathfrak{V}\,.$$

The hamiltonian function is given by the formula

$$h_\mathfrak{V} = \big\langle \widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}, \mathfrak{T}\big\rangle\,.$$

The ***pre-quantum hamiltonian for*** $h_\mathfrak{V}$, *as explicited in Definition I.3.4, is a linear operator $\widehat{\mathcal{O}}_{h_\mathfrak{V}}$ on $\Gamma\big(\mathcal{L}_{\sigma,\varnothing}\big)$ with restrictions*

$$\widehat{\mathcal{O}}_{h_\mathfrak{V}}\big|_{\pi^{-1}_{\mathsf{T}^*\mathsf{L}M}(\mathcal{O}_\mathsf{i})} = -\mathsf{i}\,\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\varnothing}}\left(\mathrm{e}^{-\theta_{\mathsf{T}^*\mathsf{L}M}}\rhd\widetilde{\mathfrak{V}}_\mathsf{i}\right)} + \big\langle \mathrm{e}^{-\theta_{\mathsf{T}^*\mathsf{L}M}} \rhd \widetilde{\mathfrak{V}}_\mathsf{i}, \mathfrak{T}\big\rangle =: \widehat{h}_{\widetilde{\mathfrak{V}}_\mathsf{i}}\,, \tag{3.17}$$

expressed in terms of local sections

$$\widetilde{\mathfrak{V}}_\mathsf{i} := \mathrm{e}^{-\pi^*_{\mathsf{T}^*\mathsf{L}M}E_\mathsf{i}} \rhd \widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V} \in \mathsf{E}^{(1,0)}_{\pi^*_{\mathsf{T}^*\mathsf{L}M}\mathcal{L}_\mathcal{G}}\mathsf{P}_{\sigma,\varnothing}\big(\pi^{-1}_{\mathsf{T}^*\mathsf{L}M}(\mathcal{O}_\mathsf{i})\big)\,. \tag{3.18}$$

Given a pair $\mathfrak{V}, \mathfrak{W}$ of σ-symmetric sections of $\mathsf{E}^{(1,1)}M$, the Poisson bracket of the associated hamiltonian functions, determined by $\Omega_{\sigma,\varnothing}$ in the manner detailed in Remark I.3.3, reads

$$\{h_\mathfrak{V}, h_\mathfrak{W}\}_{\Omega_{\sigma,\varnothing}} = h_{[\,\mathfrak{V}\,,\,\mathfrak{W}\,]^{\mathrm{H}}_{\mathrm{V}}}\,. \tag{3.19}$$

The commutator of the corresponding pre-quantum hamiltonians satisfies (locally) the relation

$$\big[\widehat{h}_{\widetilde{\mathfrak{V}}_\mathsf{i}}, \widehat{h}_{\widetilde{\mathfrak{W}}_\mathsf{i}}\big] = -\mathsf{i}\,\widehat{h}_{[\,\widetilde{\mathfrak{V}}_\mathsf{i}\,,\,\widetilde{\mathfrak{W}}_\mathsf{i}\,]_{\mathrm{V}}}\,, \tag{3.20}$$

written in terms of the bracket of the global Vinogradov structure $\mathfrak{V}^{(0)}_{\pi^*_{\mathsf{T}^*\mathsf{L}M}\mathcal{L}_\mathcal{G}}\mathsf{P}_{\sigma,\varnothing}$.

Proof. Begin by noting that the symplectic form of Eq. (3.3) can be written as

$$\Omega_{\sigma,\varnothing} = \delta\theta_{\mathsf{T}^*\mathsf{L}M} + \widetilde{\mathsf{L}}^*\mathrm{H} \equiv \delta_{\widetilde{\mathsf{L}}^*\mathrm{H}}\mathfrak{T}\,,$$

and so, using Eqs. (3.14), (3.7) and (3.8), as well as the assumption $\mathfrak{V} \in \ker \mathsf{d}_\mathrm{H}$, we find, for the canonical projection $\mathrm{pr}_{\mathsf{T}^*M} : \mathsf{E}^{(1,1)}M \to \mathsf{T}^*M$,

$$\begin{aligned}\alpha_{\mathsf{TP}_{\sigma,\varnothing}}\big(\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\big) \lrcorner\, \Omega_{\sigma,\varnothing} &= \widetilde{\mathsf{L}}_*\alpha_{\mathsf{T}M}(\mathfrak{V}) \lrcorner \big(\delta\theta_{\mathsf{T}^*\mathsf{L}M} + \widetilde{\mathsf{L}}^*\mathrm{H}\big) = -\delta\big(\widetilde{\mathsf{L}}_*\alpha_{\mathsf{T}M}(\mathfrak{V}) \lrcorner\, \theta_{\mathsf{T}^*\mathsf{L}M}\big) - \widetilde{\mathsf{L}}^*\big(\alpha_{\mathsf{T}M}(\mathfrak{V}) \lrcorner\, \mathrm{H}\big) \\ &= -\delta\big(\widetilde{\mathsf{L}}_*\alpha_{\mathsf{T}M}(\mathfrak{V}) \lrcorner\, \theta_{\mathsf{T}^*\mathsf{L}M} + \widetilde{\mathsf{L}}^*\mathrm{pr}_{\mathsf{T}^*M}(\mathfrak{V})\big)\,,\end{aligned}$$

as claimed. The form of the pre-quantum hamiltonian then follows directly from the general definition of Eq. (I.3.8), and we readily see, through direct inspection, that the local objects $\widetilde{\mathfrak{V}}_\mathsf{i}$ are in the image

of an isomorphism defined analogously to the isomorphism ${}^{\mathsf{P}_{\sigma,\varnothing}}\chi^{-1}$ from the constructive proof of Theorem 3.11.

The Poisson bracket of a pair of hamiltonian functions can be computed directly but instead let us combine our observation from Proposition 3.1 with the findings of Lemma 3.10 to render the algebraic structure that underlies the calculation manifest. First, we write down the hamiltonian section $\mathfrak{X}_{h_{\mathfrak{V}}} \equiv \widetilde{\mathfrak{V}}$ of $\mathsf{E}^{(1,0)}\mathsf{P}_{\sigma,\varnothing}$ associated to $h_{\mathfrak{V}}$. Clearly,

$$\widetilde{\mathfrak{V}} = \alpha_{\mathsf{TP}_{\sigma,\varnothing}}\big(\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\big) \oplus \big\langle\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V},\mathfrak{T}\big\rangle = \mathrm{e}^{\theta_{\mathsf{T}^*\mathsf{L}M}} \triangleright \widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\,,$$

and so, exploiting the results from the proof of Proposition 2.5 and taking into account Eq. (3.12), we find, for a pair $\mathfrak{V},\mathfrak{W} \in \Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)$,

$$\begin{aligned}\big[\widetilde{\mathfrak{V}}\,,\,\widetilde{\mathfrak{W}}\big]_{\mathrm{V}}^{\Omega_{\sigma,\varnothing}} &\equiv \big[\,\mathrm{e}^{\theta_{\mathsf{T}^*\mathsf{L}M}} \triangleright \widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\,,\,\mathrm{e}^{\theta_{\mathsf{T}^*\mathsf{L}M}} \triangleright \widetilde{\mathsf{L}}_{(1,1)}\mathfrak{W}\,\big]_{\mathrm{V}}^{\Omega_{\sigma,\varnothing}} = \mathrm{e}^{\theta_{\mathsf{T}^*\mathsf{L}M}} \triangleright \big[\,\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\,,\,\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{W}\,\big]_{\mathrm{V}}^{\widetilde{\mathsf{L}}^*\mathrm{H}}\\ &= \mathrm{e}^{\theta_{\mathsf{T}^*\mathsf{L}M}} \triangleright \widetilde{\mathsf{L}}_{(1,1)}\big([\,\mathfrak{V}\,,\,\mathfrak{W}\,]_{\mathrm{V}}^{\mathrm{H}}\big) \equiv \widetilde{[\,\mathfrak{V}\,,\,\mathfrak{W}\,]}_{\mathrm{V}}^{\mathrm{H}}\,.\end{aligned}$$

By virtue of Proposition 3.1, this confirms Eq. (3.19).

Passing, next, to the pre-quantum hamiltonians, we first note that they satisfy – in consequence of Eq. (I.3.9) – the algebra

$$\big[\widehat{h}_{\widetilde{\mathfrak{V}}_{\mathrm{i}}}\,,\widehat{h}_{\widetilde{\mathfrak{W}}_{\mathrm{i}}}\big] = -\mathsf{i}\,\widehat{\mathcal{O}}_{\{h_{\mathfrak{V}},h_{\mathfrak{W}}\}_{\Omega_{\sigma,\varnothing}}}\big|_{\mathcal{O}_{\mathrm{i}}^*}\,,$$

which can be rewritten – with the help of Eqs. (3.19) and (3.17), taken together with Eq. (3.18) – as

$$\big[\widehat{h}_{\widetilde{\mathfrak{V}}_{\mathrm{i}}}\,,\widehat{h}_{\widetilde{\mathfrak{W}}_{\mathrm{i}}}\big] = -\mathsf{i}\left(-\mathsf{i}\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\varnothing}}\left(\mathrm{e}^{-\theta_{\sigma,\varnothing\,\mathrm{i}}}\triangleright\widetilde{\mathsf{L}}_{(1,1)}[\,\mathfrak{V}\,,\mathfrak{W}\,]_{\mathrm{V}}^{\mathrm{H}}\right)} + \left\langle \mathrm{e}^{-\theta_{\sigma,\varnothing\,\mathrm{i}}} \triangleright \widetilde{\mathsf{L}}_{(1,1)}\,[\,\mathfrak{V}\,,\,\mathfrak{W}\,]_{\mathrm{V}}^{\mathrm{H}}\,,\mathfrak{T}\right\rangle\right).$$

Employing Eqs. (3.12) and (2.21) once more, we then find

$$\begin{aligned}\big[\widehat{h}_{\widetilde{\mathfrak{V}}_{\mathrm{i}}}\,,\widehat{h}_{\widetilde{\mathfrak{W}}_{\mathrm{i}}}\big] &= -\mathsf{i}\left(-\mathsf{i}\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\varnothing}}\left(\mathrm{e}^{-\theta_{\sigma,\varnothing\,\mathrm{i}}}\triangleright[\,\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\,,\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{W}\,]_{\mathrm{V}}^{\widetilde{\mathsf{L}}^*\mathrm{H}}\right)} + \left\langle \mathrm{e}^{-\theta_{\sigma,\varnothing\,\mathrm{i}}} \triangleright \big[\,\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{V}\,,\,\widetilde{\mathsf{L}}_{(1,1)}\mathfrak{W}\,\big]_{\mathrm{V}}^{\widetilde{\mathsf{L}}^*\mathrm{H}}\,,\mathfrak{T}\right\rangle\right)\\ &= -\mathsf{i}\left(-\mathsf{i}\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\varnothing}}\left(\mathrm{e}^{-\theta_{\mathsf{T}^*\mathsf{L}M}}\triangleright[\,\widetilde{\mathfrak{V}}_{\mathrm{i}}\,,\widetilde{\mathfrak{W}}_{\mathrm{i}}\,]_{\mathrm{V}}\right)} + \left\langle \mathrm{e}^{-\theta_{\mathsf{T}^*\mathsf{L}M}} \triangleright \big[\,\widetilde{\mathfrak{V}}_{\mathrm{i}}\,,\,\widetilde{\mathfrak{W}}_{\mathrm{i}}\,\big]_{\mathrm{V}}\,,\mathfrak{T}\right\rangle\right) \equiv -\mathsf{i}\,\widehat{h}_{[\,\widetilde{\mathfrak{V}}_{\mathrm{i}}\,,\widetilde{\mathfrak{W}}_{\mathrm{i}}\,]_{\mathrm{V}}}\,,\end{aligned}$$

as claimed. □

We are now ready to discuss at length the realisation of internal symmetries of the σ-model on the classical and pre-quantum state space of the untwisted sector of the theory. Thus,

Proposition 3.13. *In the notation of Corollary 3.4, of Propositions 3.5 and 3.12, and of Theorem 3.11, the σ-symmetric sections $\mathfrak{K}_A$ determine a symplectic realisation of $\mathfrak{g}_\sigma$ on $C^\infty\big(\mathsf{P}_{\sigma,\varnothing},\mathbb{R}\big)$ by the hamiltonian functions $h_{\mathfrak{K}_A}$ and an operator realisation on $\Gamma\big(\mathcal{L}_{\sigma,\varnothing}\big)$ by the pre-quantum hamiltonians $\widehat{\mathcal{O}}_{h_{\mathfrak{K}_A}}$ with local restrictions $\widehat{h}_{\widetilde{\mathfrak{K}}_A\,\mathrm{i}}$. The former realisation is hamiltonian,*

$$\{h_{\mathfrak{K}_A}, h_{\mathfrak{K}_B}\}_{\Omega_{\sigma,\varnothing}} = f_{ABC}\,h_{\mathfrak{K}_C}\,, \tag{3.21}$$

iff the $\mathfrak{K}_A$ can be chosen such that

$$\mathscr{L}_{\mathscr{K}_A}\kappa_B = f_{ABC}\,\kappa_C + \mathsf{d}D_{AB} \tag{3.22}$$

for some $D_{AB} \in C^\infty(M,\mathbb{R})$, in which case also

$$[\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]_{\mathrm{V}}^{\mathrm{H}} = f_{ABC}\,\mathfrak{K}_C + 0 \oplus \tfrac{1}{2}\,\mathsf{d}(D_{AB} - D_{BA}) \tag{3.23}$$

and

$$\big[\,\widetilde{\mathfrak{K}}_{A\,\mathrm{i}}\,,\,\widetilde{\mathfrak{K}}_{B\,\mathrm{i}}\,\big]_{\mathrm{V}} = f_{ABC}\,\widetilde{\mathfrak{K}}_{C\,\mathrm{i}}\,, \tag{3.24}$$

so that

$$\big[\widehat{h}_{\widetilde{\mathfrak{K}}_A\,\mathrm{i}}, \widehat{h}_{\widetilde{\mathfrak{K}}_B\,\mathrm{i}}\big] = -\mathsf{i}\,f_{ABC}\,\widehat{h}_{\widetilde{\mathfrak{K}}_C\,\mathrm{i}}\,. \tag{3.25}$$

Proof. First, invoking Eq. (3.5) in conjunction with Eq. (3.8), we rewrite Eq. (3.19) in the present setting as

$$\{h_{\mathfrak{K}_A}, h_{\mathfrak{K}_B}\}_{\Omega_{\sigma,\varnothing}} = f_{ABC}\,h_{\mathfrak{K}_C} + \widetilde{\mathsf{L}}^*\big(\Delta_{AB} - \mathsf{d}\mathsf{c}_{(AB)}\big) = f_{ABC}\,h_{\mathfrak{K}_C} + \widetilde{\mathsf{L}}^*\Delta_{AB}$$

in the notation of Eq. (3.5). Clearly, the realisation of $\mathfrak{g}_\sigma$ is hamiltonian iff $\Delta_{AB} = \mathsf{d}D_{AB}$ for some $D_{AB} \in C^\infty(M,\mathbb{R})$, which is, indeed, tantamount to Eq. (3.22). Moreover, note that, in this case,

$$\mathsf{d}\mathsf{c}_{(AB)} = \tfrac{1}{2}\big(\mathscr{L}_{\mathscr{K}_A}\kappa_B + \mathscr{K}_A \lrcorner \mathscr{K}_B \lrcorner \mathrm{H} + \mathscr{L}_{\mathscr{K}_B}\kappa_A + \mathscr{K}_B \lrcorner \mathscr{K}_A \lrcorner \mathrm{H}\big) = \tfrac{1}{2}\,\mathsf{d}(D_{AB} + D_{BA})\,,$$

whence Eq. (3.23) follows.

Passing to the pre-quantum hamiltonians, under the assumption that Eq. (3.22) holds true, we readily verify the identity

$$\begin{aligned}\big[\,\widetilde{\mathfrak{K}}_{A\,\mathrm{i}}\,,\,\widetilde{\mathfrak{K}}_{B\,\mathrm{i}}\,\big]_{\mathrm{V}} &= \mathrm{e}^{-\pi_{\mathsf{T}^*M}^* E_{\mathrm{i}}} \vartriangleright \widetilde{\mathsf{L}}_{(1,1)}\,[\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]_{\mathrm{V}}^{\mathrm{H}} = \mathrm{e}^{-\pi_{\mathsf{T}^*M}^* E_{\mathrm{i}}} \vartriangleright \widetilde{\mathsf{L}}_{(1,1)}\big(f_{ABC}\,\mathfrak{K}_C + 0 \oplus \tfrac{1}{2}\,\mathsf{d}(D_{AB} - D_{BA})\big)\\ &= f_{ABC}\,\widetilde{\mathfrak{K}}_{C\,\mathrm{i}}\end{aligned}$$

by reversing and repeating the manipulations carried out in the proof of Eq. (3.20), which reproduces Eq. (3.24) and thereby completes the proof. □

Remark 3.14. Another piece of evidence in favour of the relevance of the geometry of the generalised tangent bundle in the canonical description of the two-dimensional σ-model comes from the study of the Poisson algebra of the Noehter currents ($\widehat{t}$ is the normalised tangent vector field on $\mathbb{S}^1$)

$$J_{\mathfrak{K}_A}[\psi] = \mathscr{K}_A^\mu\,\mathsf{p}_\mu + (X_*\widehat{t})^\mu\,\kappa_{A\,\mu}\,, \qquad\qquad \psi = (X^\mu, \mathsf{p}_\nu) \tag{3.26}$$

of the theory, furnishing an anomalous field-theoretic representation of the algebra $\mathfrak{g}_\sigma$. This is, in fact, the structure originally examined in the pioneering Ref. [AS05] in which the link between the current-symmetry algebra of the σ-model and the structure of a Courant algebroid, twisted according to the standard prescription first suggested in Ref. [ŠW01], on the generalised tangent bundle $\mathsf{E}^{(1,1)}M$ was established. A straightforward computation, first carried out in Ref. [AS05], yields the identity

$$\{J_{\mathfrak{K}_A}(t,\varphi), J_{\mathfrak{K}_B}(t,\varphi')\}_{\Omega_{\sigma,\varnothing}} = J_{[\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]_{\mathrm{V}}^{\mathrm{H}}}(t,\varphi)\,\delta(\varphi-\varphi') - 2\,\langle\mathfrak{K}_A,\mathfrak{K}_B\rangle\,\big(t,\tfrac{1}{2}(\varphi+\varphi')\big)\,\delta'(\varphi-\varphi') \tag{3.27}$$

for the H-twisted Vinogradov bracket on $\Gamma(\mathsf{E}^{(1,1)}M)$ (identical with the Courant bracket[6] in this special case).

Remark 3.15. The contents of the present section seem to indicate that it is natural, in the context of the two-dimensional field theory of interest, to separate the algebraic structure on the generalised tangent bundle over the target space M of the σ-model from field-theoretic considerations of internal symmetries of the latter. Although the presence of a gerbe over M can affect this structure, either via the topological twist of the bundle itself or, equivalently, via the twist of the Vinogradov bracket on it, it is not a priori clear how one could extract from the structure any information on the geometry of the target. That one can actually do so was shown in Ref. [Hit06], and we pause briefly to demonstrate what can be learnt from the original argument about the transition undergone by the geometry of the tangent bundle as we pass from an untwisted to an H-twisted Vinogradov structure on the associated generalised tangent bundle of type $(1,1)$.

To these ends, we consider a target $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$, together with the generalised tangent bundle $\mathsf{E}^{(1,1)}M \to M$ and the H-twisted Vinogradov structure $\mathfrak{V}^{(1),\mathrm{H}}M$ on it, with $\mathrm{H} = H_{\lambda\mu\nu}\,\mathsf{d}X^\mu \wedge \mathsf{d}X^\mu \wedge \mathsf{d}X^\nu \in Z^3(M)$ the curvature of $\mathcal{G}$. We represent – after Hitchin – the metric g on $\Gamma(\mathsf{T}M)$ by its graph in $\mathsf{E}^{(1,1)}M$, *i.e.* by the subbundle of rank $\dim M$ with fibre

$$\mathrm{graph}(\mathrm{g})|_m := \{\ \mathscr{V}_+(m) := \mathscr{V}(m) \oplus \mathrm{g}_m(\mathscr{V}(m),\cdot) \quad | \quad \mathscr{V} \in \Gamma(\mathsf{T}M)\ \} \subset \mathsf{E}^{(1,1)}_m M\,, \qquad m \in M$$

on which the canonical contraction becomes positive definite (for g Riemannian),

$$(\,\mathscr{V}_+\,,\,\mathscr{V}_+\,)_\lrcorner = \mathrm{g}(\mathscr{V},\mathscr{V})\,.$$

Thus, we think of g as the so-called **generalised metric**, in the sense of Ref. [Hit06, Def. 3], defining a splitting

$$\mathsf{E}^{(1,1)}M = \mathrm{graph}(\mathrm{g}) \oplus \mathrm{graph}(\mathrm{g})^{\perp_{(\cdot,\cdot)_\lrcorner}}\,,$$

$$\mathscr{V} \oplus \upsilon = \tfrac{1}{2}\,\big(\mathscr{V} + \mathrm{g}^{-1}(\upsilon,\cdot)\big)_+ \oplus \tfrac{1}{2}\,\big(\mathscr{V} - \mathrm{g}^{-1}(\upsilon,\cdot)\big)_-\,,$$

where $\mathscr{W}_\pm = \mathscr{W} \oplus \mathrm{g}(\pm\mathscr{W},\cdot)$. Next, we readily check, *cf.* Ref. [Hit06, Thm. 2], that the linear operator $\nabla_{\mathscr{V}} : \Gamma(\mathsf{T}M) \to \Gamma(\mathsf{T}M)$ given, for arbitrary $\mathscr{V},\mathscr{W} \in \Gamma(\mathsf{T}M)$, by the formula

$$0 \oplus 2\mathrm{g}(\nabla_{\mathscr{V}}\mathscr{W},\cdot) := [\,\mathscr{V}_-\,,\,\mathscr{W}_+\,]_{\mathrm{V}}^{\mathrm{H}} - [\,\mathscr{V},\mathscr{W}\,]_-$$

[6] There is, in fact, a whole family of brackets on $\Gamma(\mathsf{E}^{(1,1)}M)$ of different skew-symmetry properties and jacobiators, including, in particular, the Dorfman bracket of Ref. [Dor87]. They can be obtained from the above calculation by replacing $\frac{1}{2}(\varphi+\varphi')$ in the anomalous second term on the right-hand side of Eq. (3.27) with a generic argument $\varphi_\lambda = \lambda\,\varphi + (1-\lambda)\,\varphi'$, $\lambda \in [0,1]$ and by changing the first term accordingly.

defines a metric connection

$$\nabla \quad : \quad \Gamma(\mathsf{T}M) \to \Gamma(\mathsf{T}^*M) \otimes \Gamma(\mathsf{T}M)$$

on $\mathsf{T}M$, with

$$\nabla_{\mathscr{V}} \mathscr{W} = \mathscr{V}^\lambda \left(\partial_\lambda \mathscr{W}^\nu + \Gamma_{\lambda\mu}^{\ \ \nu} \mathscr{W}^\mu \right) \partial_\nu \,,$$

where

$$\Gamma_{\lambda\mu}^{\ \ \nu} = \left\{ {}_{\lambda\mu}^{\ \nu} \right\} - 3\,(\mathrm{g}^{-1})^{\nu\rho}\,\mathrm{H}_{\rho\lambda\mu}\,.$$

Hence, the passage from the untwisted Vinogradov structure on $\mathsf{E}^{(1,1)}M$ to the twisted one can be understood in terms of induction of a torsion-full (Weitzenböck) connection on the tangent bundle of the base manifold M that extends the standard symmetric Levi-Civita connection. In this manner, $\mathfrak{V}^{(1),\mathrm{H}}M$ can encode non-trivial information on the geometry (of the tangent bundle) of the σ-model target. It deserves to be pointed out that the above identification of the curvature of the gerbe with the torsion component of a metric connection on $\mathsf{T}M$ arises independently in the framework of spectral non-commutative geometry of (the supersymmetric extension of) the CFT of the quantised σ-model, mentioned in Remark I.3.20, *cf.* Refs. [FG94, RS08].

4. Morphisms of Vinogradov structures and symmetry transmission across defects

The observations made in the preceding section suggest that we begin our study of structures induced by data carried by the defect in the geometry of the generalised tangent bundle of the σ-model background in abstraction from symmetries of the two-dimensional field theory. Thus, we find

Theorem 4.1. *Adopt the notation of Definitions 2.4 and 2.14, and of Corollary 2.17. Let $\mathfrak{B}$ be a string background with target $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$ and $\mathcal{G}$-bi-brane $\mathcal{B} = \big(Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1,2\}\big)$ as in Definition I.2.1, and let $\mathrm{H} \in Z^3(M)$ be the curvature of $\mathcal{G}$. Finally, let ${}^Q\chi_\alpha \ :\ \mathfrak{V}^{(1)}_{\iota_\alpha^*\mathcal{G}}Q \to \mathfrak{V}^{(1),\iota_\alpha^*\mathrm{H}}Q$, $\alpha \in \{1,2\}$ be the canonical isomorphisms of Corollary 2.17. Then, the following statements hold true:*

i) *The curvature ω of the $\mathcal{G}$-bi-brane $\mathcal{B}$ canonically determines an isomorphism*

$$\beta_\omega \ :\ \mathfrak{V}^{(1),\iota_1^*\mathrm{H}}Q \xrightarrow{\cong} \mathfrak{V}^{(1),\iota_2^*\mathrm{H}}Q\,.$$

ii) *The 1-isomorphism Φ of the $\mathcal{G}$-bi-brane $\mathcal{B}$ canonically induces an isomorphism*

$$\beta_\Phi \ :\ \mathfrak{V}^{(1)}_{\iota_1^*\mathcal{G}}Q \xrightarrow{\cong} \mathfrak{V}^{(1)}_{\iota_2^*\mathcal{G}}Q\,.$$

iii) *The above isomorphisms are intertwined by the isomorphisms ${}^Q\chi_\alpha$ in the sense expressed by the commutative diagram*

$$\begin{CD}
\mathfrak{V}^{(1)}_{\iota_1^*\mathcal{G}}Q @>{\beta_\Phi}>> \mathfrak{V}^{(1)}_{\iota_2^*\mathcal{G}}Q \\
@V{{}^Q\chi_1}VV @VV{{}^Q\chi_2}V \\
\mathfrak{V}^{(1),\iota_1^*\mathrm{H}}Q @>{\beta_\omega}>> \mathfrak{V}^{(1),\iota_2^*\mathrm{H}}Q
\end{CD}\,.$$

Proof. Ad i) Consider the isomorphism

$$\mathrm{e}^{-\omega} \ :\ \mathsf{E}^{(1,1)}Q \xrightarrow{\cong} \mathsf{E}^{(1,1)}Q$$

covering the identity diffeomorphism on the base Q. Reasoning as in the proof of Proposition 2.5, and using Eq. (I.3.21), we establish the identities

$$\begin{aligned}
[\,\cdot\,,\,\cdot\,]_{\mathrm{V}}^{\iota_2^*\mathrm{H}} \circ (\mathrm{e}^{-\omega}, \mathrm{e}^{-\omega}) &= \mathrm{e}^{-\omega} \circ [\,\cdot\,,\,\cdot\,]_{\mathrm{V}}^{\iota_1^*\mathrm{H}}\,, \\
(\,\cdot\,,\,\cdot\,)_\lrcorner \circ (\mathrm{e}^{-\omega}, \mathrm{e}^{-\omega}) &= (\,\cdot\,,\,\cdot\,)_\lrcorner\,, \\
\alpha_{\mathsf{T}Q} \circ \mathrm{e}^{-\omega} &= \alpha_{\mathsf{T}Q}\,.
\end{aligned}$$

This permits to set

$$\beta_\omega := \mathrm{e}^{-\omega}\,. \tag{4.1}$$

Ad ii) Choose open covers ${}^M\mathcal{O} = \{\mathcal{O}_i^M\}_{i\in\mathscr{I}_M}$ and ${}^Q\mathcal{O} = \{{}^Q\mathcal{O}_a\}_{a\in\mathscr{I}_Q}$ of the target space M and of the $\mathcal{G}$-bi-brane world-volume Q, respectively, for which there exist Čech-extended $\mathcal{G}$-bi-brane maps $(\iota_\alpha, \phi_\alpha)$, $\alpha \in \{1,2\}$, and fix local presentations, associated with these covers, for the gerbe, $(B_i, A_{ij}, g_{ijk}) \in \mathcal{A}^{3,2}({}^M\mathcal{O})$, and for the $\mathcal{G}$-bi-brane 1-isomorphism, $(P_a, K_{ab}) \in \mathcal{A}^{2,1}({}^Q\mathcal{O})$, all as described in Definition I.2.2. The transition maps $\mathfrak{g}^\alpha_{ab}$ of the $\mathsf{E}^{(1,1)}_{\iota_\alpha^*\mathcal{G}}Q$ are then given by the formula

$$\mathfrak{g}^\alpha_{ab} = \mathrm{e}^{\iota_\alpha^* B_{\phi_\alpha(a)} - \iota_\alpha^* B_{\phi_\alpha(b)}}\,.$$

We readily check, with the help of the cohomological identity from (I.2.5), that the isomorphism

$$(\mathfrak{b}_{\Phi\,a})_{a\in\mathscr{I}_Q} \;:\; \mathsf{E}^{(1,1)}_{\iota_1^*\mathcal{G}}Q \xrightarrow{\cong} \mathsf{E}^{(1,1)}_{\iota_2^*\mathcal{G}}Q\,, \qquad\qquad \mathfrak{b}_{\Phi\,a} := \mathrm{e}^{-\mathsf{d}P_a}$$

covering the identity diffeomorphism on the base Q satisfies the identity

$$\mathfrak{g}^2_{ab} = \mathfrak{b}_{\Phi\,b} \circ \mathfrak{g}^1_{ab} \circ \mathfrak{b}^{-1}_{\Phi\,a}$$

on $\mathcal{O}^Q_{ab}$, and so – by virtue of Proposition 2.13 – it is meaningful to define

$$\beta_\Phi := (\mathfrak{b}_{\Phi\,a})_{a\in\mathscr{I}_Q}\,. \tag{4.2}$$

Ad iii) An immediate corollary to Eq. (I.2.4), taking into account Eqs. (4.1) and (4.2).

□

Having established an independent interpretation of the structure of a $\mathcal{G}$-bi-brane in the context of the geometry of the generalised tangent bundle, we may next return to the main point of our interest, that is the physics of the two-dimensional σ-model in the presence of defects. In Section I.4, the latter were straightforwardly related to dualities of the σ-model. The following result attests, once more, the naturalness of the algebraic structures introduced in this paper by demonstrating their simple behaviour under dualities.

Proposition 4.2. *Adopt the notation of Definitions 2.4 and 2.14, and of Theorems 4.1 and 3.11. Let $\mathfrak{I}_\sigma(\mathcal{B})$ be the isotropic subspace in $\mathsf{P}^{\times 2}_{\sigma,\varnothing} = \mathsf{P}_{\sigma,\varnothing} \times \mathsf{P}_{\sigma,\varnothing}$ defined in Proposition I.4.1. Suppose that $\mathfrak{I}_\sigma(\mathcal{B})$ is a graph of a symplectomorphism $\beta_\mathcal{B} : \mathsf{P}_{\sigma,\varnothing} \to \mathsf{P}_{\sigma,\varnothing}$, which is the case, in particular, if the $\mathcal{G}$-bi-brane $\mathcal{B}$ together with the Defect Gluing Condition (I.2.7) define a pre-quantum duality of the untwisted sector of the σ-model, understood in the sense of Definition I.4.7. Then, $(\beta_\mathcal{B}, \widehat{\beta}_\mathcal{B})$, with the covering map*

$$\widehat{\beta}_\mathcal{B} = \begin{pmatrix} \beta_{\mathcal{B}\,*} & 0 \\ 0 & (\beta_\mathcal{B}^{-1})^* \end{pmatrix},$$

is an automorphism of $\mathfrak{V}^{(0),\Omega_{\sigma,\varnothing}}\mathsf{P}_{\sigma,\varnothing}$. Conversely, every (unital) automorphism (f, F) of $\mathfrak{V}^{(0),\Omega_{\sigma,\varnothing}}\mathsf{P}_{\sigma,\varnothing}$ is of the form

$$F = \widehat{f} \circ \mathrm{e}^{\mathrm{B}}$$

for some $\widehat{f}$ and e^{B} as in Proposition 2.5, and for B a unique 1-form on $\mathsf{P}_{\sigma,\varnothing}$ such that

$$f^*\Omega_{\sigma,\varnothing} - \Omega_{\sigma,\varnothing} = \delta\mathrm{B}\,.$$

Proof. The first statement of the proposition is readily checked through inspection, and so we pass immediately to the second one. As in the proof of Proposition 2.5, we decompose the bundle map

$$F = \widehat{f} \circ G$$

into the standard term $\widehat{f}$ that covers f, and the completion G covering the identity diffeomorphism on the base $\mathsf{P}_{\sigma,\varnothing}$. Using the identity

$$\widehat{f} \circ [\cdot\,,\cdot]_{\mathrm{V}}^{f^*\Omega_{\sigma,\varnothing}} = [\cdot\,,\cdot]_{\mathrm{V}}^{\Omega_{\sigma,\varnothing}} \circ \big(\widehat{f}, \widehat{f}\big)\,,$$

we rewrite the (co)defining property

$$[\cdot\,,\cdot]_{\mathrm{V}}^{\Omega_{\sigma,\varnothing}} \circ (F, F) = F \circ [\cdot\,,\cdot]_{\mathrm{V}}^{\Omega_{\sigma,\varnothing}} \tag{4.3}$$

of F in the form

$$G \circ [\cdot\,,\cdot]_{\mathrm{V}}^{\Omega_{\sigma,\varnothing}} = [\cdot\,,\cdot]_{\mathrm{V}}^{f^*\Omega_{\sigma,\varnothing}} \circ (G, G)\,. \tag{4.4}$$

We subsequently evaluate both sides of the last relation on the pair $(g\cdot\mathfrak{V},\mathfrak{W})$, whereby we arrive, once again, at the consistency condition (2.12) that leads to the familiar form (2.13) of G. The requirement that F be unital yields the desired result

$$G = {\rm e}^{\rm B}\,,\qquad {\rm B}\in\Omega^1(\mathsf{P}_{\sigma,\varnothing})\,.$$

Substituting the ensuing $F = \widehat{f}\circ{\rm e}^{\rm B}$ back into relation (4.3) and evaluating the latter on a pair $(\mathscr{V}\oplus g,\mathscr{W}\oplus h)$, we obtain

$$\begin{aligned}&[f_*\mathscr{V},f_*\mathscr{W}]\oplus\big[f_*\mathscr{V}\big(f^{-1\,*}(h+\mathscr{W}\lrcorner{\rm B})\big)-f_*\mathscr{W}\big(f^{-1\,*}(g+\mathscr{V}\lrcorner{\rm B})\big)+f_*\mathscr{V}\lrcorner f_*\mathscr{W}\lrcorner\Omega_{\sigma,\varnothing}\big]\\ =\quad& f_*[\mathscr{V},\mathscr{W}]\oplus f^{-1\,*}\left(\mathscr{V}(h)-\mathscr{W}(g)+\mathscr{V}\lrcorner\mathscr{W}\lrcorner\Omega_{\sigma,\varnothing}+[\mathscr{V},\mathscr{W}]\lrcorner{\rm B}\right)\,,\end{aligned}$$

whence also the consistency constraint

$$\mathscr{V}\lrcorner\mathscr{W}\lrcorner\left(f^*\Omega_{\sigma,\varnothing}-\Omega_{\sigma,\varnothing}-\delta{\rm B}\right)=0\,,$$

from which we recover the claim of the proposition. □

We are now fully equipped for the study of mechanisms of transmission of symmetries between phases of the σ-model across world-sheet defects that separate them. The obvious starting point of our analysis is a counterpart of Proposition 2.2, readily extractable from the results of Ref. [RS09], that holds true in the presence of an embedded defect quiver for $n=1$ (a generalisation of this result to higher-dimensional cases is completely straightforward).

Proposition 4.3. [RS09, App. A2] *Adopt the notation of Definition 3.2 and of Proposition 3.3. Let* $\mathscr{V}$ *be a vector field on the target space of the background* $\mathscr{F}:=\mathscr{M}\sqcup Q\sqcup\bigsqcup_{n\in\mathbb{N}_{\geq 3}}T_n$ *with a (local) flow* $\xi_t\ :\ \mathscr{F}\to\mathscr{F}$ *and restrictions* ${}^{\mathscr{M}}\mathscr{V}:=\mathscr{V}|_{\mathscr{M}}$, $\mathscr{M}\in\{M,Q,T_n\}$ *such that*

$$\iota_{\alpha\,*}{}^Q\mathscr{V}={}^M\mathscr{V}|_{\iota_\alpha(Q)}\,,\tag{4.5}$$

and

$$\pi_{n\,*}^{k,k+1\,T_n}\mathscr{V}={}^Q\mathscr{V}|_{\pi_n^{k,k+1}(T_n)}\,.$$

The variation along ξ_t *of the action functional of Eq.* (3.2) *reads*

$$\tfrac{\mathsf{d}}{\mathsf{d}t}|_{t=0}S_\sigma[(\xi_t\circ X\,|\,\Gamma);\gamma]=-\tfrac{1}{2}\int_\Sigma(\mathscr{L}_{{}^M\mathscr{V}}{\rm g})_X(\mathsf{d}X\overset{\wedge}{,}\star_\gamma\mathsf{d}X)+\int_\Sigma X^*({}^M\mathscr{V}\lrcorner{\rm H})+\int_\Gamma(X|_\Gamma)^*({}^Q\mathscr{V}\lrcorner\omega)\,.\tag{4.6}$$

Combining the above result with the statement of Corollary 3.4, we can give a compact algebraic description of internal symmetries of the σ-model in the presence of circular defects, which we formulate as

Proposition 4.4. *Adopt the notation of Definition 2.4 and of Theorem 4.1, and write*

$$\Delta_Q:=\iota_2^*-\iota_1^*\,.$$

Infinitesimal rigid symmetries of the two-dimensional non-linear σ*-model for network-field configurations* $(X\,|\,\Gamma)$ *in string background* $\mathfrak{B}$ *on world-sheet* (Σ,γ) *with a defect quiver* Γ *composed of a finite number of non-intersecting circular defect lines, as described in Definition I.2.7, correspond to pairs* $({}^M\mathfrak{V},{}^Q\mathfrak{V})$ *consisting, each, of a* σ*-symmetric section* ${}^M\mathfrak{V}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)$ *of* $\mathsf{E}^{(1,1)}M$,

$$\mathscr{L}_{\alpha_{\mathsf{T}M}({}^M\mathfrak{V})}{\rm g}=0\,,\qquad\qquad \mathsf{d}_{\rm H}{}^M\mathfrak{V}=0\,,\tag{4.7}$$

and of a ${}^M\mathfrak{V}$*-twisted* σ*-symmetric section* ${}^Q\mathfrak{V}\in\Gamma\big(\mathsf{E}^{(1,0)}Q\big)$ *of* $\mathsf{E}^{(1,0)}Q$,

$$\mathsf{d}_\omega{}^Q\mathfrak{V}=-\Delta_Q{\rm pr}_{\mathsf{T}^*M}({}^M\mathfrak{V})\,,\tag{4.8}$$

written in terms of the canonical projection ${\rm pr}_{\mathsf{T}^*M}:\mathsf{E}^{(1,1)}M\to\mathsf{T}^*M$, *and subject to the* ι_α***-alignment condition:***

$$\alpha_{\mathsf{T}M}({}^M\mathfrak{V})|_{\iota_\alpha(Q)}=\iota_{\alpha\,*}\alpha_{\mathsf{T}Q}({}^Q\mathfrak{V})\,.\tag{4.9}$$

Proof. The sufficiency of the conditions listed is a straightforward corollary to Proposition 4.3. That they are also necessary was demonstrated in the proof of Proposition 2.24 of Ref. [GSW12]. □

There are some important consequences of the statement of symmetry of the σ-model in the presence of defects. We begin with

Proposition 4.5. *In the notation of Definition 2.4, and of Theorems 4.1 and 3.11, the Poisson(-bracket) algebra of the hamiltonian functions on* $\mathsf{P}_{\sigma,\varnothing}$ *assigned to those* σ*-symmetric sections of* $\mathsf{E}^{(1,1)}M$ *which admit an extension to a pair of* σ*-symmetric sections from* $\mathsf{E}^{(1,1)}M \sqcup \mathsf{E}^{(1,0)}Q$ *subject to the* ι_α*-alignment condition* (4.9) *is continuous across* Γ.

Proof. Take an arbitrary pair of ι_α-aligned σ-symmetric sections $\big({}^M\mathscr{V} \oplus \upsilon, {}^Q\mathscr{V} \oplus \xi\big) =: ({}^M\mathfrak{V}, {}^Q\mathfrak{V})$. The proof boils down to demonstrating the equality of the values $h_{{}^M\mathfrak{V}}[\psi_1]$ and $h_{{}^M\mathfrak{V}}[\psi_2]$ attained by the hamiltonian function $h_{{}^M\mathfrak{V}}$ on a pair $(\psi_1, \psi_2) = \big((X_1, \mathsf{p}_1), (X_2, \mathsf{p}_2)\big)$ of untwisted states from the isotropic subspace $\mathfrak{I}_\sigma(\mathcal{B})$ introduced in Proposition I.4.1. We obtain, in the notation adopted from the proof of that proposition, and using Eqs. (I.2.7) and (4.8),

$$\begin{aligned}
h_{{}^M\mathfrak{V}}[\psi_1] &= \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1)\,\big({}^M\mathscr{V}(X_1) \lrcorner\, \mathsf{p}_1 + (X_{1\,*}\widehat{t}) \lrcorner\, \upsilon(X_1)\big) \\
&= \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1)\,\big({}^Q\mathscr{V}(X) \lrcorner\, (\mathsf{p}_1 \circ \iota_{1\,*}) + (X_*\widehat{t}) \lrcorner\, \iota_1^*\upsilon(X)\big) \\
&= \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1)\,\big({}^Q\mathscr{V}(X) \lrcorner\, (\mathsf{p}_2 \circ \iota_{2\,*}) + (X_*\widehat{t}) \lrcorner\, (\iota_1^*\upsilon - {}^Q\mathscr{V} \lrcorner\, \omega)(X)\big) \\
&= \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1)\,\big({}^Q\mathscr{V}(X) \lrcorner\, (\mathsf{p}_2 \circ \iota_{2\,*}) + (X_*\widehat{t}) \lrcorner\, (\iota_2^*\upsilon + \mathsf{d}\xi)(X)\big) \\
&= \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1)\,\big({}^M\mathscr{V}(X_2) \lrcorner\, \mathsf{p}_2 + (X_{2\,*}\widehat{t}) \lrcorner\, \upsilon(X_2)\big) \\
&= h_{{}^M\mathfrak{V}}[\psi_2]\,.
\end{aligned}$$

This is manifestly consistent with the structure of the Poisson algebra on the space of hamiltonian functions. □

The last result carries over directly to the pre-quantum régime, in which we have the analogous

Proposition 4.6. *Adopt the notation of Definition 2.4, and of Theorems 4.1 and 3.11, and assume that the isotropic submanifold* $\mathfrak{I}_\sigma(\mathcal{B}) \subset \mathsf{P}_{\sigma,\varnothing} \times \mathsf{P}_{\sigma,\varnothing}$ *defined in Proposition I.4.1 is a graph of a symplectomorphism. The unitary similarity transformation on the set of pre-quantum hamiltonians of the untwisted sector of the* σ*-model defined by the bundle isomorphism* $\mathfrak{D}_\sigma(\mathcal{B})$ *from the proof of Theorem I.4.9 preserves (element-wise) the subalgebra composed of those pre-quantum hamiltonians which are assigned to the* σ*-symmetric sections of* $\mathsf{E}^{(1,1)}M$ *admitting an extension to a pair of* σ*-symmetric sections from* $\mathsf{E}^{(1,1)}M \sqcup \mathsf{E}^{(1,0)}Q$ *subject to the* ι_α*-alignment condition* (4.9).

Proof. Fix an open cover $\mathcal{O}_{\mathfrak{I}_\sigma(\mathcal{B})} = \{\mathcal{O}^*_{\mathrm{i}^1} \times \mathcal{O}^*_{\mathrm{i}^2}\}_{\mathscr{I}_{\mathfrak{I}_\sigma(\mathcal{B})}}$ of $\mathfrak{I}_\sigma(\mathcal{B})$ as in the proof of Theorem I.4.9 and take the associated data $\mathrm{pr}_\alpha^*(\theta_{\sigma,\varnothing\,\mathrm{i}^\alpha}, \gamma_{\sigma,\varnothing\,\mathrm{i}^\alpha\mathrm{j}^\alpha})$, $\alpha \in \{1,2\}$ of the pullbacks $\mathrm{pr}_\alpha^*\mathcal{L}_{\sigma,\varnothing}$ of the pre-quantum bundle $\mathcal{L}_{\sigma,\varnothing} \to \mathsf{P}_{\sigma,\varnothing}$ from the same corollary, and those of the bundle isomorphism $\mathfrak{D}_\sigma(\mathcal{B})$, denoted by $f_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2)}$ and given in Eq. (I.4.12). The latter relate local sections $\mathrm{pr}_\alpha^* s_{\mathrm{i}^\alpha} : \mathcal{O}^*_{\mathrm{i}^1} \times \mathcal{O}^*_{\mathrm{i}^2} \to \mathrm{pr}_\alpha^*\mathcal{L}_{\sigma,\varnothing}$ over $\mathfrak{I}_\sigma(\mathcal{B}) \ni (\psi_1, \psi_2)$ as per

$$s_{\mathrm{i}^2}[\psi_2] = f_{\sigma,\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \cdot s_{\mathrm{i}^1}[\psi_1]\,.$$

Take, next, a section $\mathfrak{V} = \mathscr{V} \oplus \upsilon \in \Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)$ and consider the associated local pre-quantum hamiltonians $\widehat{h}_{\mathfrak{V}_\mathrm{i}}$ from Eq. (3.17), written out explicitly as

$$\widehat{h}_{\mathfrak{V}_\mathrm{i}} = -\mathrm{i}\,\mathscr{L}_{\widetilde{\mathsf{L}_*\mathscr{V}}} - \widetilde{\mathsf{L}_*\mathscr{V}} \lrcorner\, \theta_{\sigma,\varnothing\,\mathrm{i}} + h_{\mathfrak{V}}\,,$$

cf. Eq. (I.3.8). Upon invoking continuity of the hamiltonian function $h_{\mathfrak{V}}$ across the defect, demonstrated in the proof of Proposition 4.5, and using relation (I.4.10), we then find

$$\begin{aligned}
\widehat{h}_{\mathfrak{V}_{\mathrm{i}^2}}[\psi_2] \triangleright s_{\mathrm{i}^2}[\psi_2] &\equiv \big(-\mathrm{i}\,\mathscr{L}_{\widetilde{\mathsf{L}_*\mathscr{V}[\psi_2]}}|_{\psi_1=\mathrm{const}} - \widetilde{\mathsf{L}_*\mathscr{V}} \lrcorner\, \theta_{\sigma,\varnothing\,\mathrm{i}^2}[\psi_2] + h_{\mathfrak{V}}[\psi_2]\big) s_{\mathrm{i}^2}[\psi_2] \\
&= \big(-\mathrm{i}\,\mathscr{L}_{\widetilde{\mathsf{L}_*\mathscr{V}[\psi_2]}}|_{\psi_1=\mathrm{const}} - \mathrm{i}\,\mathscr{L}_{\widetilde{\mathsf{L}_*\mathscr{V}[\psi_1]}}|_{\psi_2=\mathrm{const}} - \widetilde{\mathsf{L}_*\mathscr{V}} \lrcorner\, \theta_{\sigma,\varnothing\,\mathrm{i}^2}[\psi_2] + h_{\mathfrak{V}}[\psi_2]\big) s_{\mathrm{i}^2}[\psi_2] \\
&= f_{\sigma,\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \cdot \big(-\mathrm{i}\,\mathscr{L}_{\widetilde{\mathsf{L}_*\mathscr{V}[\psi_1]}}|_{\psi_2=\mathrm{const}} - \widetilde{\mathsf{L}_*\mathscr{V}} \lrcorner\, \theta_{\sigma,\varnothing\,\mathrm{i}^2}[\psi_2] + h_{\mathfrak{V}}[\psi_1] \\
&\quad - \big(\widetilde{\mathsf{L}_*\mathscr{V}[\psi_2]}|_{\psi_1=\mathrm{const}} + \widetilde{\mathsf{L}_*\mathscr{V}[\psi_1]}|_{\psi_2=\mathrm{const}}\big) \lrcorner\, \mathrm{i}\,\delta \log f_{\sigma,\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)]\big) s_{\mathrm{i}^1}[\psi_1] \\
&= f_{\sigma,\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \cdot \big(-\mathrm{i}\,\mathscr{L}_{\widetilde{\mathsf{L}_*\mathscr{V}[\psi_1]}}|_{\psi_2=\mathrm{const}} - \widetilde{\mathsf{L}_*\mathscr{V}} \lrcorner\, \theta_{\sigma,\varnothing\,\mathrm{i}^1}[\psi_1] + h_{\mathfrak{V}}[\psi_1]\big) s_{\mathrm{i}^1}[\psi_1]
\end{aligned}$$

$$\equiv \quad f_{\sigma,\mathcal{B}(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)]\cdot\big(\widehat{h}_{\widetilde{\mathfrak{V}}_{\mathrm{i}^1}}[\psi_1]\triangleright s_{\mathrm{i}^1}[\psi_1]\big)\,,$$

as claimed. □

The present section rendered more precise the intuitively clear assignment, to the geometric data carried by the defect, of morphisms in the category of (twisted) Vinogradov structures on the (twisted) generalised tangent bundle over the target space of the background, the objects of the latter category being viewed as target-space counterparts of the canonical Vinogradov structure on the state space of the untwisted sector of the σ-model, naturally associated with symmetries of that sector. It also clarified the conditions under which symmetries of the theory are mapped to one another across the defect on the level of the corresponding hamiltonian functions and pre-quantum hamiltonians, and – in so doing – pointed towards an extension of the previous category that would accommodate the ι_α-aligned σ-symmetric sections. The natural question as to the precise nature of this extension becomes particularly pronounced when discussing a realisation of the transmitted symmetries in the twisted sector of the theory, which we examine closely in the next section.

5. Paired bracket structures and symmetries of the twisted sector

The emergence of the distinguished gerbe bi-modules associated with bi-branes follows a natural pattern of cohomological, or – more abstractly – categorial descent, laid out in Ref. [Ste00] and further elaborated in Ref. [FNSW09] and similar in spirit to the one discussed in Remark I.5.6, in which a lower-rank cohomological structure arises from trivialisation of a pullback-cohomology[7] coboundary obtained by pulling back a higher-rank structure to a correspondence space along a number of smooth maps between the bases of the geometric objects corresponding to the two structures. In the process, the classifying cohomology for the lower-rank structure inherits a twist, *cf.* Eq. (I.2.4), which couples the two structures together. Drawing inspiration from the intimate relationship between (n-)gerbes and bracket structures on generalised tangent bundles, noted in Ref. [Hit03, Gua03] and further elaborated in the preceding sections, we propose to follow the same line of reasoning in the algebraic setting of generalised geometry. In so doing, we use the principle of compatibility with the symmetry content of the two-dimensional σ-model as a natural measure of naturalness of our constructions. We are thus led to the following

Definition 5.1. Let (M,Q) be a pair of smooth manifolds, equipped with a pair of smooth maps $\iota_\alpha : Q\to M,\ \alpha\in\{1,2\}$ and a pair $(\mathrm{H},\omega)\in\Omega^3(M)\times\Omega^2(Q)$ of globally defined forms. Write $\Delta_Q=\iota_2^*-\iota_1^*$ and assume

$$\mathsf{d}\omega+\Delta_Q\mathrm{H}=0\,. \tag{5.1}$$

Adopt the notation of Definitions 2.4 and 2.14 and let $\mathrm{pr}_{\mathsf{T}^*M}:\mathsf{E}^{(1,1)}M\to\mathsf{T}^*M$ and $\mathrm{pr}_{\mathsf{T}^*Q}:\mathsf{E}^{(1,0)}Q\to\mathsf{T}^*Q$ be the canonical projections. The **$(\mathrm{H},\omega;\Delta_Q)$-twisted bracket structure on ι_α-paired generalised tangent bundles** $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\to M\sqcup Q$ is the quadruple

$$\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q,[\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)},(\cdot,\cdot)_\lrcorner,\alpha_{\mathsf{T}(M\sqcup Q)}\big)=:\mathfrak{M}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}(M\sqcup Q)$$

in which $(\cdot,\cdot)_\lrcorner$ and $\alpha_{\mathsf{T}(M\sqcup Q)}$ restrict to the respective canonical contractions and anchors on the component generalised tangent bundles, and in which $[\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)}$ is the antisymmetric bilinear operation on smooth sections of $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$ that assigns to a pair $\mathfrak{V},\mathfrak{W}\in\Gamma\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ of sections, with restrictions $\mathfrak{V}|_{\mathscr{M}}={}^{\mathscr{M}}\mathfrak{V},\ \mathfrak{W}|_{\mathscr{M}}={}^{\mathscr{M}}\mathfrak{W},\ \mathscr{M}\in\{M,Q\}$, another section with restrictions

$$[\![\mathfrak{V},\mathfrak{W}]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_M \quad=\quad \big[{}^M\mathfrak{V},{}^M\mathfrak{W}\big]^{\mathrm{H}}_{\mathrm{V}}\,,$$

$$[\![\mathfrak{V},\mathfrak{W}]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_Q \quad=\quad \big[{}^Q\mathfrak{V},{}^Q\mathfrak{W}\big]^{\omega}_{\mathrm{V}}+0\oplus\tfrac{1}{2}\big(\alpha_{\mathsf{T}Q}({}^Q\mathfrak{V})\lrcorner\Delta_Q\mathrm{pr}_{\mathsf{T}^*M}({}^M\mathfrak{W})-\alpha_{\mathsf{T}Q}({}^Q\mathfrak{W})\lrcorner\Delta_Q\mathrm{pr}_{\mathsf{T}^*M}({}^M\mathfrak{V})\big)\,.$$

Given two such structures, $\mathfrak{M}^{(1,0),(\mathrm{H}_n,\omega_n;\Delta_{Q_n})}(M_n\sqcup Q_n),\ n\in\{1,2\}$, on the respective ι^n_α-paired generalised tangent bundles $\mathsf{E}^{(1,1)}M_n\sqcup\mathsf{E}^{(1,0)}Q_n\to M_n\sqcup Q_n$, a **(factorised[8]) homomorphism between twisted bracket structures on paired generalised tangent bundles** is a quadruple

[7] *Cf.* Ref. [Mur96].

[8] We could, in principle, contemplate more general mappings, mixing sections of the two pairs of generalised tangent bundles involved.

$\big(f^{(1)},F^{(1,1)},f^{(0)},F^{(1,0)}\big)$ which consists of a pair of diffeomorphisms[9]

$$f^{(1)}\ :\ M_1\to M_2\,,\qquad\qquad f^{(0)}\ :\ Q_1\to Q_2$$

compatible with the ι_α^n in the sense expressed by the commutative diagram

$$\begin{tikzcd} Q_1 \arrow[r, "f^{(0)}"] \arrow[d, "\iota_\alpha^1"'] & Q_2 \arrow[d, "\iota_\alpha^2"] \\ M_1 \arrow[r, "f^{(1)}"] & M_2 \end{tikzcd}\,,$$

together with the vector-bundle maps

$$F^{(1,1)}\ :\ \mathsf{E}^{(1,1)}M_1\to\mathsf{E}^{(1,1)}M_2\,,\qquad\qquad F^{(1,0)}\ :\ \mathsf{E}^{(1,0)}Q_1\to\mathsf{E}^{(1,0)}Q_2$$

that cover $f^{(1)}$ and $f^{(0)}$, respectively, in the sense expressed by the commutative diagrams

$$\begin{tikzcd} \mathsf{E}^{(1,1)}M_1 \arrow[r, "F^{(1,1)}"] \arrow[d, "\pi_{\mathsf{T}M_1}\circ\alpha_{\mathsf{T}M_1}"'] & \mathsf{E}^{(1,1)}M_2 \arrow[d, "\pi_{\mathsf{T}M_2}\circ\alpha_{\mathsf{T}M_2}"] \\ M_1 \arrow[r, "f^{(1)}"] & M_2 \end{tikzcd}\,,\qquad \begin{tikzcd} \mathsf{E}^{(1,0)}Q_1 \arrow[r, "F^{(1,0)}"] \arrow[d, "\pi_{\mathsf{T}Q_1}\circ\alpha_{\mathsf{T}Q_1}"'] & \mathsf{E}^{(1,0)}Q_2 \arrow[d, "\pi_{\mathsf{T}Q_2}\circ\alpha_{\mathsf{T}Q_2}"] \\ Q_1 \arrow[r, "f^{(0)}"] & Q_2 \end{tikzcd}\,,$$

and such that the following identities hold true for $F^{(1,1\sqcup 0)}=F^{(1,1)}\sqcup F^{(1,0)}$:

$$[\![\cdot,\cdot]\!]^{(\mathrm{H}_2,\omega_2;\Delta_{Q_2})}\circ\big(F^{(1,1\sqcup 0)},F^{(1,1\sqcup 0)}\big)\ =\ F^{(1,1\sqcup 0)}\circ[\![\cdot,\cdot]\!]^{(\mathrm{H}_1,\omega_1;\Delta_{Q_1})}\,,\tag{5.2}$$

$$(\cdot,\cdot)_\lrcorner\circ\big(F^{(1,1\sqcup 0)},F^{(1,1\sqcup 0)}\big)\ =\ \Big(\big(f^{(1)\,-1}\big)^*\sqcup\big(f^{(0)\,-1}\big)^*\Big)\circ(\cdot,\cdot)_\lrcorner\,,\tag{5.3}$$

$$\alpha_{\mathsf{T}(M_2\sqcup Q_2)}\circ F^{(1,1\sqcup 0)}\ =\ \big(f^{(1)}_*\sqcup f^{(0)}_*\big)\circ\alpha_{\mathsf{T}(M_1\sqcup Q_1)}\,.\tag{5.4}$$

In analogy with Proposition 2.5, we readily prove

Proposition 5.2. *Adopt the notation of Definitions 2.4 and 5.1, and suppose that* $\big(f^{(1)},F^{(1,1)},f^{(0)},F^{(1,0)}\big)$ *is an automorphism of the* $(\mathrm{H},\omega;\Delta_Q)$*-twisted bracket structure on* ι_α*-paired generalised tangent bundles* $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$*. Then,* $\big(f^{(1)},F^{(1,1)},f^{(0)},F^{(1,0)}\big)$ *necessarily has the following properties:*

i) *the base maps* $f^{(1)}$ *and* $f^{(0)}$ *are diffeomorphisms such that*

$$f^{(1)\,*}\mathrm{H}-\mathrm{H}=\mathsf{d}\mathrm{B}^{(1)}\,,\qquad\qquad f^{(0)\,*}\omega-\omega=\mathsf{d}\mathrm{B}^{(0)}\tag{5.5}$$

for some $\mathrm{B}^{(1)}\in\Omega^2(M)$ *and* $\mathrm{B}^{(0)}\in\Omega^1(Q)$*, of which the former is further constrained by the condition*

$$\Delta_Q\big({}^M\mathscr{V}\lrcorner\,\mathrm{B}^{(1)}\big)=0\,,\tag{5.6}$$

to be satisfied for an arbitrary vector field ${}^M\mathscr{V}$ *on* M*;*

ii) *the bundle maps take the form*

$$F^{(1,1)}\sqcup F^{(1,0)}=\Big(\widehat{f}^{(1)}\circ\mathrm{e}^{\mathrm{B}^{(1)}}\Big)\sqcup\Big(\widehat{f}^{(0)}\circ\mathrm{e}^{\mathrm{B}^{(0)}}\Big)\,.\tag{5.7}$$

Upon restriction to the subspace $\Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)\subset\Gamma\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ *composed of those sections, to be termed* ι_α***-aligned****, which satisfy the additional condition*

$$\iota_{\alpha\,*}\circ\alpha_{\mathsf{T}Q}=\alpha_{\mathsf{T}M}\big|_{\iota_\alpha(Q)}\,,\tag{5.8}$$

the set of automorphisms extends to include those with base maps constrained as in the first of Eqs. (5.5)*, and with bundle maps as in Eq.* (5.7) *but now written for forms* $\mathrm{B}^{(1)}$ *and* $\mathrm{B}^{(0)}$ *subject to the constraint*

$$\mathsf{d}\mathrm{B}^{(0)}=f^{(0)\,*}\omega-\omega+\Delta_Q\mathrm{B}^{(1)}\,.\tag{5.9}$$

---

[9]Clearly, one could relax the requirement that the base maps be diffeomorphisms, whereupon a notion of a morphism of the two brackets would be obtained. Here, we consider the more rigid structure with view to the subsequent physical applications.

Proof. The proof goes along similar lines as that of Proposition 4.2, which is also how the form of the bundle map $F^{(1,1)}$ is established. Only now one considers an automorphism $G := \widehat{f^{(1)}}^{-1} \circ F^{(1,1)}$ of $\mathsf{E}^{(1,1)}M$ satisfying the analogon of relation (4.4).

The sole non-trivial statement that has to be verified is the one concerning the explicit form of the bundle map $F^{(1,0)}$. We begin by noting that condition (5.4) fixes the map in the form

$$F^{(1,0)} = \begin{pmatrix} \mathrm{id}_{\mathsf{T}Q} & 0 \\ \\ \mathrm{B}^{(0)} & C \end{pmatrix}$$

for some $\mathrm{B}^{(0)} \in \Gamma(\mathsf{T}^*Q)$ and $C \in C^\infty(Q,\mathbb{R})$. Due to the triviality of condition (5.3), we are left with condition (5.2) to be imposed. Take arbitrary sections $\mathfrak{V},\mathfrak{W} \in \Gamma\big(\mathsf{E}^{(1,1)}M \sqcup \mathsf{E}^{(1,0)}Q\big)$ with restrictions $(\mathfrak{V},\mathfrak{W})|_M = ({}^M\mathscr{V} \oplus \upsilon, {}^M\mathscr{W} \oplus \varpi)$ and $(\mathfrak{V},\mathfrak{W})|_Q = ({}^Q\mathscr{V} \oplus \xi, {}^Q\mathscr{W} \oplus \zeta)$. The condition now boils down to the identity

$$\begin{aligned}
&[{}^Q\mathscr{V}, {}^Q\mathscr{W}] \lrcorner \mathrm{B}^{(0)} + C \cdot \big({}^Q\mathscr{V} \lrcorner \mathsf{d}\zeta - {}^Q\mathscr{W} \lrcorner \mathsf{d}\xi + {}^Q\mathscr{V} \lrcorner {}^Q\mathscr{W} \lrcorner \omega + \tfrac{1}{2}\big({}^Q\mathscr{V} \lrcorner \Delta_Q\varpi - {}^Q\mathscr{W} \lrcorner \Delta_Q\upsilon\big)\big) \\
= \;& {}^Q\mathscr{V} \lrcorner \mathsf{d}\big({}^Q\mathscr{W} \lrcorner \mathrm{B}^{(0)} + C \cdot \zeta\big) - {}^Q\mathscr{W} \lrcorner \mathsf{d}\big({}^Q\mathscr{V} \lrcorner \mathrm{B}^{(0)} + C \cdot \xi\big) + {}^Q\mathscr{V} \lrcorner {}^Q\mathscr{W} \lrcorner f^{(0)\,*}\omega \\
&+ \tfrac{1}{2}\big({}^Q\mathscr{V} \lrcorner \Delta_Q\big(\varpi + {}^M\mathscr{W} \lrcorner \mathrm{B}^{(1)}\big) - {}^Q\mathscr{W} \lrcorner \Delta_Q\big(\upsilon + {}^M\mathscr{V} \lrcorner \mathrm{B}^{(1)}\big)\big),
\end{aligned}$$

or – after obvious cancellations –

$$\begin{aligned}
&{}^Q\mathscr{W} \lrcorner {}^Q\mathscr{V} \lrcorner \big(\mathsf{d}\mathrm{B}^{(0)} + (C - f^{(0)\,*})\,\omega\big) + (\zeta\,{}^Q\mathscr{V} - \xi\,{}^Q\mathscr{W}) \lrcorner \mathsf{d}C \\
= \;& \tfrac{1}{2}\big[{}^Q\mathscr{W} \lrcorner \big(\Delta_Q\big(\upsilon + {}^M\mathscr{V} \lrcorner \mathrm{B}^{(1)}\big) - C\,\Delta_Q\upsilon\big) - {}^Q\mathscr{V} \lrcorner \big(\Delta_Q\big(\varpi + {}^M\mathscr{W} \lrcorner \mathrm{B}^{(1)}\big) - C\,\Delta_Q\varpi\big)\big].
\end{aligned}$$

On setting ${}^Q\mathscr{W} = -{}^Q\mathscr{V}$, ${}^M\mathscr{W} = -{}^M\mathscr{V}$ and $\varpi = -\upsilon$, the above simplifies as

$$(\xi + \zeta)\,{}^Q\mathscr{V} \lrcorner \mathsf{d}C = 0\,,$$

whence $C \in \mathbb{R}$. Keeping the same relation between the vector components but letting υ and ϖ vary independently, we fix the value of the constant as $C = 1$ (for $\iota_1 \not\equiv \iota_2$, which we assume). This leaves us with the condition

$${}^Q\mathscr{W} \lrcorner {}^Q\mathscr{V} \lrcorner \big(\mathsf{d}\mathrm{B}^{(0)} + (1 - f^{(0)\,*})\omega\big) + \tfrac{1}{2}\big({}^Q\mathscr{V} \lrcorner \Delta_Q\big({}^M\mathscr{W} \lrcorner \mathrm{B}^{(1)}\big) - {}^Q\mathscr{W} \lrcorner \Delta_Q\big({}^M\mathscr{V} \lrcorner \mathrm{B}^{(1)}\big)\big) = 0\,. \qquad (5.10)$$

Up to now, the special choices made along the way were always consistent with the additional constraint (5.8), and so differentiation between generic automorphisms and the extended ones for the restricted bracket structure starts at this point.

In order to ultimately constrain the former, set ${}^M\mathscr{V} = 0 = {}^M\mathscr{W}$ to obtain the second of Eqs. (5.5). The ensuing constraint

$${}^Q\mathscr{V} \lrcorner \Delta_Q\big({}^M\mathscr{W} \lrcorner \mathrm{B}^{(1)}\big) - {}^Q\mathscr{W} \lrcorner \Delta_Q\big({}^M\mathscr{V} \lrcorner \mathrm{B}^{(1)}\big) = 0$$

is then tantamount to Eq. (5.6), which proves the first part of the proposition.

As for ι_α-aligned sections, note, first of all, that the restriction makes sense as

$$\begin{aligned}
\alpha_{\mathsf{T}M}\big([\![\,\mathfrak{V}, \mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}\big)|_{\iota_\alpha(Q)} &= [\alpha_{\mathsf{T}M}(\mathfrak{V}), \alpha_{\mathsf{T}M}(\mathfrak{W})]|_{\iota_\alpha(Q)} = [\iota_{\alpha\,*} \circ \alpha_{\mathsf{T}Q}(\mathfrak{V}), \iota_{\alpha\,*} \circ \alpha_{\mathsf{T}Q}(\mathfrak{W})] \\
&= \iota_{\alpha\,*} \circ [\alpha_{\mathsf{T}Q}(\mathfrak{V}), \alpha_{\mathsf{T}Q}(\mathfrak{W})] = \iota_{\alpha\,*} \circ \alpha_{\mathsf{T}Q}\big([\![\,\mathfrak{V}, \mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}\big)\,.
\end{aligned}$$

Upon restriction, Eq. (5.10) rewrites as

$${}^Q\mathscr{V} \lrcorner {}^Q\mathscr{W} \lrcorner \big(\mathsf{d}\mathrm{B}^{(0)} + (1 - f^{(0)\,*})\omega - \Delta_Q\mathrm{B}^{(1)}\big) = 0\,,$$

and so Eq. (5.9) is reproduced. This completes the proof of the proposition. □

The constraint (5.8) is completely natural in the physical context of our analysis as it is directly built into the structure of the σ-model for world-sheets with an embedded defect quiver, *cf.* Eq. (4.5). That it is also distinguished from a purely geometric point of view is shown in the following

Proposition 5.3. *Adopt the notation of Definitions 2.4 and 5.1, and of Theorem 4.1. Choose open covers* ${}^M\mathcal{O} = \{{}^M\mathcal{O}_i\}_{i \in \mathscr{I}_M}$ *and* ${}^Q\mathcal{O} = \{{}^Q\mathcal{O}_a\}_{a \in \mathscr{I}_Q}$ *such that there exist Čech extensions* $\check{\iota}_\alpha = (\iota_\alpha, \phi_\alpha)$ *of the* $\mathcal{G}$*-bi-brane maps as in Definition I.2.2. Let* $\mathsf{E}^{(1,1)}_{\mathcal{G}}M \to M$ *be the* $\mathcal{G}$*-twisted generalised tangent bundle associated with* ${}^M\mathcal{O}$*, and let* $\mathsf{E}^{(1,0)}_{\mathcal{B}}Q := \mathsf{E}^{(1,0)}_{\{\mathfrak{g}_{ab}\}}Q \to Q$ *be the generalised tangent bundle twisted*

by a local presentation $(P_a,K_{ab})\in\mathcal{A}^{2,1}({}^Q\mathcal{O})$ *of* Φ *associated, in the manner specified in Definition I.2.2, with* ${}^Q\mathcal{O}$*, with the twist determined by the transition maps*

$$\mathfrak{g}_{ab}=\mathrm{e}^{(P_a-P_b)|_{{}^Q\mathcal{O}_{ab}}}\,.$$

Write

$$\check{\Delta}_Q:=\check{\iota}_2^*-\check{\iota}_1^*\,.$$

A ***global (twisted) bracket structure***

$$\mathfrak{M}^{(1,0),(\cdot,\cdot;\Delta_Q)}_{(\mathcal{G},\mathcal{B})}(M\sqcup Q)=\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q,[\![\,\cdot\,,\,\cdot\,]\!]^{(\cdot,\cdot;\check{\Delta}_Q)},(\,\cdot\,,\,\cdot\,)_{\lrcorner},\alpha_{\mathsf{T}(M\sqcup Q)}\big)$$

on $(\mathcal{G},\mathcal{B})$***-twisted*** ι_α***-paired generalised tangent bundles*** $\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q$ *(understood in analogy with the global Vinogradov structure of Definition 2.12) exists, in general, exclusively on the subspace of* ι_α*-aligned sections* $\Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)\subset\Gamma\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)$*. The restricted bracket structure* $\mathfrak{M}^{(1,0),(0,0;\check{\Delta}_Q)}(M\sqcup Q)|_{\Gamma_{\iota_\alpha}\left(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\right)}=:\mathfrak{M}^{(1,0),(0,0;\check{\Delta}_Q)}_{(\mathcal{G},\mathcal{B}),\iota_\alpha}(M\sqcup Q)$ *is homomorphic with the restricted* $(\mathrm{H},\omega;\Delta_Q)$*-twisted bracket structure* $\mathfrak{M}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}(M\sqcup Q)|_{\Gamma_{\iota_\alpha}\left(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\right)}=:\mathfrak{M}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}_{\iota_\alpha}(M\sqcup Q)$*, and the homomorphism*

$${}^{M\sqcup Q}\chi\ :\ \mathfrak{M}^{(1,0),(0,0;\check{\Delta}_Q)}_{(\mathcal{G},\mathcal{B}),\iota_\alpha}(M\sqcup Q)\to\mathfrak{M}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}_{\iota_\alpha}(M\sqcup Q)$$

restricts as

$${}^{M\sqcup Q}\chi\ :\ \mathsf{E}^{(1,1)}_{\mathcal{G}}M\xrightarrow{\cong}\mathsf{E}^{(1,1)}M\,,\qquad\qquad {}^{M\sqcup Q}\chi\ :\ \mathsf{E}^{(1,0)}_{\mathcal{B}}Q\xrightarrow{\cong}\mathsf{E}^{(1,0)}Q$$

with local data

$${}^{M\sqcup Q}\chi|_{{}^M\mathcal{O}_i}=\mathrm{e}^{B_i}\,,\qquad\qquad {}^{M\sqcup Q}\chi|_{{}^Q\mathcal{O}_a}=\mathrm{e}^{P_a}$$

determined by a local presentation of $\mathcal{B}$ *as above and that of the gerbe,* $(B_i,A_{ij},g_{ijk})\in\mathcal{A}^{3,2}({}^M\mathcal{O})$*.*

Proof. In virtue of Corollary 2.17, and due to the triviality of the canonical contraction on $\mathsf{E}^{(1,0)}Q$, the proof of the existence of a global (twisted) bracket structure on $\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q$ reduces to checking the required properties of the bracket $[\![\,\cdot\,,\,\cdot\,]\!]^{(\cdot,\cdot;\check{\Delta}_Q)}$ restricted to Q. Choose open covers ${}^M\mathcal{O}$ and ${}^Q\mathcal{O}$ as described, and take the associated local presentation of $\mathfrak{B}$. Given a pair $\mathfrak{V},\mathfrak{W}$ of sections of $\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q$, with restrictions $(\mathfrak{V},\mathfrak{W})|_{{}^M\mathcal{O}_i}=({}^M\mathscr{V}\oplus\upsilon_i,{}^M\mathscr{W}\oplus\varpi_i)$ and $(\mathfrak{V},\mathfrak{W})|_{{}^Q\mathcal{O}_a}=({}^Q\mathscr{V}\oplus\xi_a,{}^Q\mathscr{W}\oplus\zeta_a)$, we readily compute, using Eq. (I.2.4) and for $[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\cdot,\cdot;\check{\Delta}_Q)}_a=[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\cdot,\cdot;\check{\Delta}_Q)}|_{{}^Q\mathcal{O}_a}$,

$$\big([\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\cdot,\cdot;\check{\Delta}_Q)}_b-[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\cdot,\cdot;\check{\Delta}_Q)}_a\big)_{{}^Q\mathcal{O}_{ab}}=0\oplus\Delta_{ab}$$

with

$$\begin{aligned}
\Delta_{ab}\ &=\ \big[{}^Q\mathscr{V}\lrcorner\,\mathsf{d}(\zeta_b-\zeta_a)-{}^Q\mathscr{W}\lrcorner\,\mathsf{d}(\xi_b-\xi_a)+\tfrac{1}{2}\,{}^Q\mathscr{V}\lrcorner\big(\iota_2^*(\varpi_{\phi_2(b)}-\varpi_{\phi_2(a)})-\iota_1^*(\varpi_{\phi_1(b)}-\varpi_{\phi_1(i)})\big)\\
&\quad-\tfrac{1}{2}\,{}^Q\mathscr{W}\lrcorner\big(\iota_2^*(\upsilon_{\phi_2(b)}-\upsilon_{\phi_2(a)})-\iota_1^*(\upsilon_{\phi_1(b)}-\upsilon_{\phi_1(a)})\big)\big]|_{{}^Q\mathcal{O}_{ab}}\\
&=\ \big\{{}^Q\mathscr{V}\lrcorner\,\mathsf{d}\big({}^Q\mathscr{W}\lrcorner(P_a-P_b)\big)-{}^Q\mathscr{W}\lrcorner\,\mathsf{d}\big({}^Q\mathscr{V}\lrcorner(P_a-P_b)\big)\\
&\quad+\tfrac{1}{2}\,{}^Q\mathscr{V}\lrcorner\big[\iota_2^*\big({}^M\mathscr{W}\lrcorner(B_{\phi_2(a)}-B_{\phi_2(b)})\big)-\iota_1^*\big({}^M\mathscr{W}\lrcorner(B_{\phi_1(a)}-B_{\phi_1(b)})\big)\big]\\
&\quad-\tfrac{1}{2}\,{}^Q\mathscr{W}\lrcorner\big[\iota_2^*\big({}^M\mathscr{V}\lrcorner(B_{\phi_2(a)}-B_{\phi_2(b)})\big)-\iota_1^*\big({}^M\mathscr{V}\lrcorner(B_{\phi_1(a)}-B_{\phi_1(b)})\big)\big]\big\}|_{{}^Q\mathcal{O}_{ab}}\\
&=\ [{}^Q\mathscr{V},{}^Q\mathscr{W}]\lrcorner(P_a-P_b)|_{{}^Q\mathcal{O}_{ab}}+{}^Q\mathscr{V}\lrcorner\,{}^Q\mathscr{W}\lrcorner\big(\iota_2^*\mathsf{d}A_{\phi_2(a)\phi_2(b)}-\iota_1^*\mathsf{d}A_{\phi_1(a)\phi_1(b)}\big)\\
&\quad-\tfrac{1}{2}\,\big\{{}^Q\mathscr{V}\lrcorner\big[\iota_2^*\big({}^M\mathscr{W}\lrcorner\,\mathsf{d}A_{\phi_2(a)\phi_2(b)}\big)-\iota_1^*\big({}^M\mathscr{W}\lrcorner\,\mathsf{d}A_{\phi_1(a)\phi_1(b)}\big)\big]\\
&\quad-{}^Q\mathscr{W}\lrcorner\big[\iota_2^*\big({}^M\mathscr{V}\lrcorner\,\mathsf{d}A_{\phi_2(a)\phi_2(b)}\big)-\iota_1^*\big({}^M\mathscr{V}\lrcorner\,\mathsf{d}A_{\phi_1(a)\phi_1(b)}\big)\big]\big\}\,.
\end{aligned}$$

The first term in the above expression has the desired form, and it is immediately clear that the condition for the other terms to cancel out (generically) coincides with the defining relation (5.8).

Passing to the second statement of the proposition, we see once more that it remains to prove it for the bracket restricted to Q. Thus, we have to show, for any two sections $\mathfrak{V},\mathfrak{W}$ of $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$, with restrictions $(\mathfrak{V},\mathfrak{W})|_{\mathscr{M}}=({}^{\mathscr{M}}\mathfrak{V},{}^{\mathscr{M}}\mathfrak{W})$, $\mathscr{M}\in\{M,Q\}$, the equality

$$\begin{aligned}&\big[\,\mathrm{e}^{-P_a}\triangleright{}^Q\mathfrak{V}\,,\,\mathrm{e}^{-P_a}\triangleright{}^Q\mathfrak{W}\,\big]_{\mathrm{V}}+0\oplus\tfrac{1}{2}\big[\alpha_{\mathsf{T}Q}\big(\mathrm{e}^{-P_a}\triangleright{}^Q\mathfrak{V}\big)\lrcorner\big(\iota_2^*\mathrm{pr}_{\mathsf{T}^*M}\big(\mathrm{e}^{-B_{\phi_2(a)}}\triangleright{}^M\mathfrak{W}\big)-\iota_1^*\mathrm{pr}_{\mathsf{T}^*M}\big(\mathrm{e}^{-B_{\phi_1(a)}}\triangleright{}^M\mathfrak{W}\big)\big)\\&\qquad-\alpha_{\mathsf{T}Q}\big(\mathrm{e}^{-P_a}\triangleright{}^Q\mathfrak{W}\big)\lrcorner\big(\iota_2^*\mathrm{pr}_{\mathsf{T}^*M}\big(\mathrm{e}^{-B_{\phi_2(a)}}\triangleright{}^M\mathfrak{V}\big)-\iota_1^*\mathrm{pr}_{\mathsf{T}^*M}\big(\mathrm{e}^{-B_{\phi_1(a)}}\triangleright{}^M\mathfrak{V}\big)\big)\big]\\&=\mathrm{e}^{-P_a}\triangleright\big[\big[\,{}^Q\mathfrak{V}\,,\,{}^Q\mathfrak{W}\,\big]^{\omega}_{\mathrm{V}}+0\oplus\tfrac{1}{2}\big(\alpha_{\mathsf{T}Q}\big({}^Q\mathfrak{V}\big)\lrcorner\Delta_Q\mathrm{pr}_{\mathsf{T}^*M}\big({}^M\mathfrak{W}\big)-\alpha_{\mathsf{T}Q}\big({}^Q\mathfrak{W}\big)\lrcorner\Delta_Q\mathrm{pr}_{\mathsf{T}^*M}\big({}^M\mathfrak{V}\big)\big)\big].\end{aligned}$$

It is verified through a straightforward calculation employing exactly the same arguments as those invoked in the proof of the first part of the proposition. □

Remark 5.4. It is perhaps worth emphasising at this stage that the very definition of the $(\mathcal{G},\mathcal{B})$-twisted ι_α-paired generalised tangent bundles $\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q$ ensures the existence of an isomorphism between any two such bundles twisted by (gauge-)equivalent local presentations of $\mathfrak{B}$, and this property is inherited by the global (twisted) bracket structure under the restriction.

The distinguished ι_α-aligned sections reappear in the algebraic description of symmetries of the σ-model, which we give in

Proposition 5.5. *Adopt the notation of Definitions 2.4 and 5.1, of Theorem 4.1, and of Propositions 4.4 and 5.3. Infinitesimal rigid symmetries of the two-dimensional non-linear σ-model for network-field configurations $(X\,|\,\Gamma)$ in string background $\mathfrak{B}$ on world-sheet (Σ,γ) with a defect quiver Γ composed of a finite number of non-intersecting circular defect lines, as described in Definition I.2.7, correspond to those ι_α-aligned sections of $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$ which satisfy conditions (4.7) and (4.8). We shall call them **σ-symmetric ι_α-aligned sections of** $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$, and denote the corresponding subset in $\Gamma\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ as $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$. The bracket $[\![\,\cdot\,,\,\cdot\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}$ closes on $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$,*

$$\mathfrak{V},\mathfrak{W}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)\qquad\Longrightarrow\qquad[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)\,,$$

and every bracket with this property differs from $[\![\,\cdot\,,\,\cdot\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}$ by a linear map on $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)\wedge\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ with values given by pairs $(\mathrm{B}_1,\mathrm{B}_0)\in\Omega^1(M)\times C^\infty(Q,\mathbb{R})$ subject to the constraints

$$\mathsf{d}\mathrm{B}_1=0\,,\qquad\qquad\mathsf{d}\mathrm{B}_0+\Delta_Q\mathrm{B}_1=0\,.\tag{5.11}$$

Equivalently, the symmetries can be represented by σ-symmetric ι_α-aligned sections $\mathfrak{V}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)$ of $\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q$, with restrictions $\mathfrak{V}|_{\mathscr{M}\mathcal{O}_i}={}^{\mathscr{M}}\mathfrak{V}_i$, $i\in\mathscr{I}_{\mathscr{M}}$, $\mathscr{M}\in\{M,Q\}$,

$$\mathscr{L}_{\alpha_{\mathsf{T}M}({}^M\mathfrak{V}_i)}\mathrm{g}=0\,,\qquad\qquad\left\{\begin{array}{l}\mathsf{d}\mathrm{pr}_{\mathsf{T}^*M}({}^M\mathfrak{V}_i)+\mathscr{L}_{\alpha_{\mathsf{T}M}({}^M\mathfrak{V}_i)}B_i=0\\[1em]\mathsf{d}\mathrm{pr}_{\mathsf{T}^*Q}({}^Q\mathfrak{V}_a)+\mathscr{L}_{\alpha_{\mathsf{T}Q}({}^Q\mathfrak{V}_a)}P_a=-\check{\Delta}_Q\mathrm{pr}_{\mathsf{T}^*M}({}^M\mathfrak{V}_\bullet)_a\end{array}\right..$$

The bracket $[\![\,\cdot\,,\,\cdot\,]\!]^{(0,0;\check{\Delta}_Q)}$ closes on $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)$,

$$\mathfrak{V},\mathfrak{W}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)\qquad\Longrightarrow\qquad[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(0,0;\check{\Delta}_Q)}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)\,,$$

and every bracket with this property differs from $[\![\,\cdot\,,\,\cdot\,]\!]^{(0,0;\check{\Delta}_Q)}$ by a linear map on $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)\wedge\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)$ with local values $(\mathrm{B}_{1,i},\mathrm{B}_{0,a})\in\Omega^1({}^M\mathcal{O}_i)\times C^\infty({}^Q\mathcal{O}_a,\mathbb{R})$ constrained as in Eq. (5.11).

Proof. The correspondence between symmetries and σ-symmetric ι_α-aligned sections of $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$ was established in Proposition 4.4. In the light of Corollary 3.4, it remains to verify the relation

$$\mathsf{d}_\omega[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_Q+\Delta_Q\mathrm{pr}_{\mathsf{T}^*M}\big([\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_M\big)=0\,.$$

Write $({}^M\mathfrak{V},{}^M\mathfrak{W})=({}^M\mathscr{V}\oplus\upsilon,{}^M\mathscr{W}\oplus\varpi)$ and $({}^Q\mathfrak{V},{}^Q\mathfrak{W})=({}^Q\mathscr{V}\oplus\xi,{}^Q\mathscr{W}\oplus\zeta)$. Using condition (4.5) alongside the assumption $\mathfrak{V},\mathfrak{W}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$, we obtain

$$\mathsf{d}\big({}^Q\mathscr{V}\lrcorner\mathsf{d}\zeta-{}^Q\mathscr{W}\lrcorner\mathsf{d}\xi+{}^Q\mathscr{V}\lrcorner{}^Q\mathscr{W}\lrcorner\omega+\tfrac{1}{2}\big({}^Q\mathscr{V}\lrcorner\Delta_Q\varpi-{}^Q\mathscr{W}\lrcorner\Delta_Q\upsilon\big)\big)+\big[{}^Q\mathscr{V},{}^Q\mathscr{W}\big]\lrcorner\omega$$

$$+\Delta_Q\big(\mathscr{L}_{{}^M\mathscr{V}}\varpi-\mathscr{L}_{{}^M\mathscr{W}}\upsilon-\tfrac{1}{2}\,\mathsf{d}\big({}^M\mathscr{V}\lrcorner\varpi-{}^M\mathscr{W}\lrcorner\upsilon\big)+{}^M\mathscr{V}\lrcorner{}^M\mathscr{W}\lrcorner\mathrm{H}\big)$$

$$=\quad -\mathscr{L}_{{}^Q\mathscr{V}}\big({}^Q\mathscr{W}\lrcorner\omega\big)+\mathscr{L}_{{}^Q\mathscr{W}}\big({}^Q\mathscr{V}\lrcorner\omega\big)+\mathsf{d}\big({}^Q\mathscr{V}\lrcorner{}^Q\mathscr{W}\lrcorner\omega\big)+\big[{}^Q\mathscr{V},{}^Q\mathscr{W}\big]\lrcorner\omega-{}^Q\mathscr{V}\lrcorner{}^Q\mathscr{W}\lrcorner\mathsf{d}\omega=0\,,$$

as claimed. The uniqueness of the bracket up to a bilinear map on $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)\wedge\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ with values described in the thesis of the proposition is obvious. Finally, the relations defining σ-symmetric ι_α-aligned sections of $\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q$ rephrase those defining σ-symmetric ι_α-aligned sections of $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$, and the closure of the corresponding bracket follows from Proposition 5.3. $\square$

The reconstruction of the algebraic structure present on the set of those symmetries of the untwisted sector of the σ-model that are transmitted across a conformal defect is an obvious prerequisite for understanding their symplectic realisation on the *twisted* sector of the theory, with the twist determined by the geometric data carried by the defect. In order to attain this goal, we should first lift the structures obtained hitherto on the target space and the bi-brane world-volume to the twisted loop spaces $\mathsf{L}_{Q|\{(P_k,\varepsilon_k)\}}$ and to the respective cotangent bundles. For the sake of concretness and brevity, we restrict our analysis to the 1-twisted case.

Definition 5.6. Let (M,Q) be a pair of smooth manifolds, equipped with a pair of smooth maps $\iota_\alpha:Q\to M,\ \alpha\in\{1,2\}$, and let $\mathsf{L}_{Q|(\pi,\varepsilon)}M$ be the 1-twisted loop space with coordinates (X,q), as introduced in Definition I.3.10. Write $\mathbb{S}^1_\pi:=\mathbb{S}^1\smallsetminus\{\pi\}$ for $\pi\in\mathbb{S}^1$, and denote by $\mathrm{ev}_{M,\pi}\ :\ \mathsf{L}_{Q|(\pi,\varepsilon)}M\times\mathbb{S}^1_\pi\to M$ the canonical evaluation map. A pair $({}^M\mathscr{V},{}^Q\mathscr{V})\in\Gamma(\mathsf{T}M)\times\Gamma(\mathsf{T}Q)$ of vector fields will be called ι_α**-aligned** iff

$$\iota_{\alpha*}{}^Q\mathscr{V}={}^M\mathscr{V}\big|_{\iota_\alpha(Q)}\,,$$

and the corresponding subset in $\Gamma(\mathsf{T}M)\times\Gamma(\mathsf{T}Q)$ will be denoted as $\Gamma_{\iota_\alpha}(\mathsf{T}M\sqcup\mathsf{T}Q)$. The (local) flow $\xi_t={}^M\xi_t\sqcup{}^Q\xi_t:\mathscr{M}\sqcup Q\to\mathscr{M}\sqcup Q$ of a ι_α-aligned pair $({}^M\mathscr{V},{}^Q\mathscr{V})$ (assumed to exist) satisfies the condition

$$\iota_\alpha\circ{}^Q\xi_t={}^M\xi_t\big|_{\iota_\alpha(Q)}\,.$$

The **1-twisted loop-space lift of** ι_α**-aligned pair of vector fields on** $M\sqcup Q$ is a linear map

$$\mathsf{L}^{Q|(\pi,\varepsilon)}_{\iota_\alpha *}\ :\ \Gamma_{\iota_\alpha}(\mathsf{T}M\sqcup\mathsf{T}Q)\to\Gamma(\mathsf{T}\mathsf{L}_{Q|(\pi,\varepsilon)}M)\ :\ ({}^M\mathscr{V},{}^Q\mathscr{V})\mapsto\big(\mathsf{L}^\pi_*{}^M\mathscr{V},{}^Q\mathscr{V}\circ\mathrm{pr}_Q\big)=:\mathsf{L}^{Q|(\pi,\varepsilon)}_*({}^M\mathscr{V},{}^Q\mathscr{V})\,,\tag{5.12}$$

written in terms of a loop-space lift L^π_* determined just as the loop-space lift L_* in Definition 3.6 (*i.e.* through action on functionals of 1-twisted loops) and of the canonical projection $\mathrm{pr}_Q\ :\ \mathsf{L}_{Q|(\pi,\varepsilon)}M\to Q$, so that

$${}^Q\mathscr{V}\circ\mathrm{pr}_Q[(X,q)]={}^Q\mathscr{V}(q)\,.$$

The **1-twisted loop-space lift of** n**-form from** M is a linear map

$$\mathsf{L}^{Q|(\pi,\varepsilon)\,*}\ :\ \Omega^n(M)\to\Omega^{n-1}(\mathsf{L}_{Q|(\pi,\varepsilon)}M)\ :\ \upsilon\mapsto\int_{\mathbb{S}^1_\pi}\mathrm{ev}^*_{M,\pi}\upsilon=:\mathsf{L}^{Q|(\pi,\varepsilon)\,*}\upsilon\,,\quad n\in\mathbb{N}_{>0}\,,$$

extended to the case of $n=0$ as per

$$\mathsf{L}^{Q|(\pi,\varepsilon)\,*}\ :\ C^\infty(M,\mathbb{R})\to\{0\}\ :\ f\mapsto 0\,.$$

Similarly, the **1-twisted loop-space lift of** n**-form from** Q is a linear map

$$\underline{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\ :\ \Omega^n(Q)\to\Omega^n(\mathsf{L}_{Q|(\pi,\varepsilon)}M)\ :\ \upsilon\mapsto\mathrm{pr}_Q^*\upsilon\,,\quad n\in\mathbb{N}\,.$$

The lifts thus defined can, in turn, be combined into a lift

$$\mathsf{L}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\ :\ \Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)\to\Gamma\big(\mathsf{E}^{(1,0)}\mathsf{L}_{Q|(\pi,\varepsilon)}M\big)$$

with restrictions

$$\mathsf{L}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\big|_{\Gamma_{\iota_\alpha}(\mathsf{T}M\sqcup\mathsf{T}Q)}:=\mathsf{L}^{Q|(\pi,\varepsilon)}_{\iota_\alpha *}$$

and

$$\mathsf{L}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\big|_{\Gamma(\mathsf{T}^*M\sqcup(Q\times\mathbb{R}))}:=\mathsf{L}^{Q|(\pi,\varepsilon)\,*}\circ\mathrm{pr}_{\Gamma(\mathsf{T}^*M)}+\varepsilon\,\underline{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\circ\mathrm{pr}_{C^\infty(Q,\mathbb{R})}\,,$$

written in terms of the canonical projections $\mathrm{pr}_{\Gamma(\mathsf{T}^*M)}\ :\ \Gamma\left(\mathsf{T}^*M\sqcup(Q\times\mathbb{R})\right)\to\Gamma(\mathsf{T}^*M)$ and $\mathrm{pr}_{\Gamma(\mathsf{T}^*M)}\ :\ \Gamma\left(\mathsf{T}^*M\sqcup(Q\times\mathbb{R})\right)\to C^\infty(Q,\mathbb{R})$.

By way of preparation for the subsequent discussion, we give

Lemma 5.7. *Adopt the notation of Definitions 2.4 and 5.6, Theorem 4.1 and Proposition 5.2. Let* $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ *be the 1-twisted state space with the canonical projections* $\mathrm{pr}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M} : \mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)} \to \mathsf{L}_{Q|(\pi,\varepsilon)}M$ *and* $\mathrm{pr}_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M} : \mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)} \to \mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M$*, all as introduced in Definition I.3.10. Denote by* $\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}$ *the canonical 1-form on* $\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M$ *given in that definition. The lifts* $\mathsf{L}^{Q|(\pi,\varepsilon)\,*}$ *and* $\underline{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}$ *induce the respective lifts*

$$\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*} := \mathrm{pr}^*_{\mathsf{L}_{Q|(\pi,\varepsilon)}M} \circ \mathsf{L}^{Q|(\pi,\varepsilon)\,*}\,, \qquad \widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*} := \mathrm{pr}^*_{\mathsf{L}_{Q|(\pi,\varepsilon)}M} \circ \underline{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\,,$$

and, analogously, the lift $\underline{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}$ *induces a canonical lift*

$$\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*} \;:\; \Gamma_{\iota_\alpha}(\mathsf{T}M \sqcup \mathsf{T}Q) \to \Gamma(\mathsf{T}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)})$$

fixed by the relations

$$\mathrm{pr}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M\,*} \circ \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*} = \mathsf{L}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\,, \tag{5.13}$$

$$\mathscr{L}_{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}({}^M\mathscr{V},{}^Q\mathscr{V})}\mathrm{pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M} = 0\,, \tag{5.14}$$

to be satisfied for any $({}^M\mathscr{V},{}^Q\mathscr{V}) \in \Gamma_{\iota_\alpha}(\mathsf{T}M \sqcup \mathsf{T}Q)$*. The above can, in turn, be combined into a lift*

$$\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)} \;:\; \Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}M \sqcup \mathsf{E}^{(1,0)}Q\big) \to \Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\big)$$

with restrictions

$$\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\big|_{\Gamma_{\iota_\alpha}(\mathsf{T}M\sqcup\mathsf{T}Q)} := \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*} \tag{5.15}$$

and

$$\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\big|_{\Gamma(\mathsf{T}^*M\sqcup(Q\times\mathbb{R}))} := \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*} \circ \mathrm{pr}_{\Omega^1(M)} + \varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*} \circ \mathrm{pr}_{C^\infty(Q,\mathbb{R})}\,. \tag{5.16}$$

The various lifts have the following properties

$$\delta\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\upsilon = -\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\mathsf{d}\upsilon + \varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\Delta_Q\upsilon\,, \tag{5.17}$$

$$\delta\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\xi = \widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\mathsf{d}\xi\,, \tag{5.18}$$

$$\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big) \lrcorner\, \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\upsilon = -\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\big({}^M\mathscr{V} \lrcorner\, \upsilon\big)\,, \tag{5.19}$$

$$\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big) \lrcorner\, \widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\xi = \widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\big({}^Q\mathscr{V} \lrcorner\, \xi\big)\,, \tag{5.20}$$

written for arbitrary $({}^M\mathscr{V},{}^Q\mathscr{V}) \in \Gamma_{\iota_\alpha}(\mathsf{T}M \sqcup \mathsf{T}Q), \upsilon \in \Omega^n(M)$ *and* $\xi \in \Omega^n(Q)$*.*

Proof. Obvious, through inspection. □

The next result establishes the sought-after connection between the (twisted) bracket structure on $(\mathcal{G},\mathcal{B})$-twisted ι_α-paired generalised tangent bundles $\mathsf{E}^{(1,1)}_{\mathcal{G}}M \sqcup \mathsf{E}^{(1,0)}_{\mathcal{B}}Q$ and the canonical Vinogradov structure on the (1-)twisted state space of the σ-model in the presence of defects, thus realising the general correspondence scheme anticipated in the Introduction.

Theorem 5.8. *Adopt the notation of Corollary 2.17, of Theorem 4.1, of Proposition 5.3, and of Lemma 5.7. Let* $\mathcal{L}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)} \to \mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ *be the pre-quantum bundle from Corollary I.3.19. The pair* $(\mathcal{G},\mathcal{B})$ *canonically induces a linear mapping*

$$\phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)} \;:\; \Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M \sqcup \mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big) \to \Gamma\big(\mathsf{E}^{(1,0)}_{\mathcal{L}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\big)$$

that relates elements of the respective global structures $\mathfrak{M}^{(1,0),(0,0;\check{\Delta}_Q)}_{(\mathcal{G},\mathcal{B}),\iota_\alpha}(M \sqcup Q)$ *and* $\mathfrak{V}^{(0)}_{\mathcal{L}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ *as*

$$\alpha_{\mathsf{T}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}} \circ \phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)} = \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*} \circ \alpha_{\mathsf{T}(M\sqcup Q)}\,, \tag{5.21}$$

$$[\,\cdot\,,\,\cdot\,]_{\mathrm{V}} \circ \big(\phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}, \phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\big) = \phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)} \circ [\![\,\cdot\,,\,\cdot\,]\!]^{(0,0;\check{\Delta}_Q)}\,, \tag{5.22}$$

$$(\,\cdot\,,\,\cdot\,)_{\lrcorner} \circ \big(\phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}, \phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\big) = \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*} \circ \mathrm{pr}_{\Omega^0(M)} \circ (\,\cdot\,,\,\cdot\,)_{\lrcorner} \equiv 0\,. \tag{5.23}$$

Proof. Consider the generalised tangent bundles $\mathsf{E}^{(1,1)}M$ and $\mathsf{E}^{(1,0)}Q$ in keeping with Definition 2.4. Denote by $\mathrm{H}\in Z^3(M)$ the curvature of $\mathcal{G}$, and let $\mathfrak{M}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}_{\iota_\alpha}(M\sqcup Q)$ be the $(\mathrm{H},\omega;\Delta_Q)$-twisted bracket structure on ι_α-paired generalised tangent bundles $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$, as described in Definition 5.1, under restriction to the subset $\Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ of ι_α-aligned sections of $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$, introduced in Proposition 5.2. In virtue of Proposition 5.3, there exists a homomorphism of bracket structures

$$ {}^{M\sqcup Q}\chi\ :\ \mathfrak{M}^{(1,0),(0,0;\check\Delta_Q)}_{(\mathcal{G},\mathcal{B}),\iota_\alpha}(M\sqcup Q)\to\mathfrak{M}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}_{\iota_\alpha}(M\sqcup Q)\,. $$

Consider, next, the generalised tangent bundle $\mathsf{E}^{(1,0)}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ equipped with the $\Omega_{\sigma,\mathcal{B}|(\pi,\varepsilon)}$-twisted Vinogradov structure $\mathfrak{V}^{(0),\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$, detailed in Definition 2.14. Corollary 2.17 states the existence of a homomorphisms of the Vinogradov structures

$$ {}^{\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\chi\ :\ \mathfrak{V}^{(0)}_{\mathcal{L}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\to\mathfrak{V}^{(0),\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\,, $$

given in terms of local data of $\mathcal{L}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$, *cf.* the proof of Theorem 3.11. The linear mapping announced in the theorem is now explicitly defined as

$$ \phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}:={}^{\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\chi^{-1}\circ\mathrm{e}^{\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\circ\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup0)}\circ{}^{M\sqcup Q}\chi $$

in terms of the lift $\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup0)}$ from Lemma 5.7. The linearity of the mapping thus defined follows immediately from condition (5.8) of ι_α-alignment as the latter enforces a common scaling of the two restrictions (to M and to Q) of a section from $\Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}_{\mathcal{G}}M\sqcup\mathsf{E}^{(1,0)}_{\mathcal{B}}Q\big)$. Moreover, relation (5.23) is satisfied automatically due to – on one hand – the identity

$$ \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\circ\mathrm{pr}_{\Omega^0(M)}\equiv0\,, $$

cf. Eq. (3.6), and – on the other hand – the triviality of the canonical contraction on $\mathsf{E}^{(1,0)}_{\mathcal{L}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$. This leaves us with the other two relations to check.

The first of the two, Eq. (5.21), derives directly from the definition of $\phi_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ in which all mappings except for the lift leave the vector-field components unchanged, and in which the lift itself restricts to vector-field components as in Eq. (5.15). In order to prove the other one, Eq. (5.22), it suffices to demonstrate the identity

$$ [\,\cdot\,,\,\cdot\,]_{\mathrm{V}}^{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\mathrm{H}+\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega}\circ\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup0)},\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup0)}\big)=\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup0)}\circ[\![\,\cdot\,,\,\cdot\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}\,. \tag{5.24} $$

Take a pair of sections $\mathfrak{V},\mathfrak{W}\in\Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ and denote the respective restrictions to M and Q as $\mathfrak{V}|_M={}^M\mathscr{V}\oplus\upsilon$, $\mathfrak{V}|_Q={}^Q\mathscr{V}\oplus\xi$ and $\mathfrak{W}|_M={}^M\mathscr{W}\oplus\varpi$, $\mathfrak{W}|_Q={}^Q\mathscr{W}\oplus\zeta$. Furthermore, for the sake of transparency, represent $\mathfrak{V}$ as $\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\oplus(\upsilon,\xi)$, and $\mathfrak{W}$ as $\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\oplus(\varpi,\zeta)$, and similarly for their bracket. Upon invoking conditions (5.13) and (5.14) in conjunction with Eq. (5.12), condition (5.8), and Eqs. (5.19) together with (5.17), and (5.20) together with (5.18), this yields

$$ \begin{aligned} \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup0)}\circ[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)} &= \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big([{}^M\mathscr{V},{}^M\mathscr{W}],[{}^Q\mathscr{V}\,,\,{}^Q\mathscr{W}]\big)\\ &\quad\oplus\Big[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\big(\mathscr{L}_{{}^M\mathscr{V}}\varpi-\mathscr{L}_{{}^M\mathscr{W}}\upsilon-\tfrac{1}{2}\,\mathsf{d}\big({}^M\mathscr{V}\lrcorner\,\varpi-{}^M\mathscr{W}\lrcorner\,\upsilon\big)+{}^M\mathscr{V}\lrcorner\,{}^M\mathscr{W}\lrcorner\,\mathrm{H}\big)\\ &\quad+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\big({}^Q\mathscr{V}\lrcorner\,\mathsf{d}\zeta-{}^Q\mathscr{W}\lrcorner\,\mathsf{d}\xi+{}^Q\mathscr{V}\lrcorner\,{}^Q\mathscr{W}\lrcorner\,\omega+\tfrac{1}{2}\,\big({}^Q\mathscr{V}\lrcorner\,\Delta_Q\varpi-{}^Q\mathscr{W}\lrcorner\,\Delta_Q\upsilon\big)\big)\Big]\\ &= \big[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big),\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\big]\\ &\quad\oplus\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\delta\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\varpi-\tfrac{\varepsilon}{2}\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\Delta_Q\varpi\\ &\quad-\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\lrcorner\,\delta\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\upsilon+\tfrac{\varepsilon}{2}\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\lrcorner\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\Delta_Q\upsilon\\ &\quad+\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\lrcorner\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\mathrm{H}\\ &\quad+\varepsilon\,\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\delta\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\zeta-\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\lrcorner\,\delta\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\xi\big)\\ &\quad+\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\lrcorner\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega\\ &\quad+\tfrac{\varepsilon}{2}\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\big({}^Q\mathscr{V}\lrcorner\,\Delta_Q\varpi-{}^Q\mathscr{W}\lrcorner\,\Delta_Q\upsilon\big)\big) \end{aligned} $$

$$
\begin{aligned}
&= \big[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big),\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\big]\\
&\quad\oplus\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\delta\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\varpi+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\zeta\big)\\
&\quad-\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\lrcorner\,\delta\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\upsilon+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\xi\big)\\
&\quad+\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\lrcorner\,\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\mathrm{H}+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega\big)\,.
\end{aligned}
$$

Comparison with Eq. (2.5) and, subsequently, with Eqs. (5.16) and (5.15) permits to rewrite the above concisely as

$$
\begin{aligned}
\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\circ[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)} &= \Big[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\oplus\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\upsilon+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\xi\big)\,,\\
&\qquad\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{W},{}^Q\mathscr{W}\big)\oplus\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\varpi+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\zeta\big)\Big]_{\mathrm{V}}^{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\mathrm{H}+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega}\\
&= \Big[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{1,1\sqcup 0}\mathfrak{V}\,,\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{1,1\sqcup 0}\mathfrak{W}\Big]_{\mathrm{V}}^{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\mathrm{H}+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega}\,,
\end{aligned}
$$

which concludes the proof. □

We are now fully equipped to discuss, in the algebraic framework elaborated above, a realisation of symmetries in the twisted sector of the theory.

Proposition 5.9. *Adopt the notation of Definitions 2.4 and 5.6, of Corollary 2.17, of Theorems 4.1 and 5.8, of Propositions 5.2 and 5.3, and of Lemma 5.7. Let* $\mathcal{L}_{(\mathcal{G},\mathcal{B})|(\pi,\varepsilon)}\to\mathsf{L}_{Q|(\pi,\varepsilon)}M$ *be the transgression bundle of Theorem I.3.18, with local data* $\big(E_{(\pi,\varepsilon)\,\mathrm{i}},G_{(\pi,\varepsilon)\,\mathrm{ij}}\big)$*, as explicited in the constructive proof of the theorem, written for the open cover* $\mathcal{O}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}=\{\mathcal{O}_{(\pi,\varepsilon)\,\mathrm{i}}\}_{\mathrm{i}\in\mathscr{I}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}}$ *of* $\mathsf{L}_{Q|(\pi,\varepsilon)}$ *from Proposition I.3.14. Write*

$$
\mathfrak{T}_{\mathcal{B}\,|\,(\pi,\varepsilon)}:=1\oplus\mathrm{pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\in\Gamma\big(\mathsf{E}^{(0,1)}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\big)\,,
$$

and call the latter object the ***canonical section of*** $\mathsf{E}^{(1,0)}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$*. To every σ-symmetric ι_α-aligned section* $\mathfrak{V}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ *there is associated a* ***hamiltonian function*** $h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{V}}$*, i.e. a smooth function on* $\mathsf{P}_{\sigma,\mathcal{B}|(\pi,\varepsilon)}$ *satisfying the defining relation*

$$
\alpha_{\mathsf{TP}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\big)\lrcorner\,\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}=:-\delta h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{V}}\,.
$$

The hamiltonian function is given by the formula

$$
h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{V}}=\Big\langle\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V},\mathfrak{T}_{\mathcal{B}\,|\,(\pi,\varepsilon)}\Big\rangle\,.\tag{5.25}
$$

The ***pre-quantum hamiltonian for*** $h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{V}}$*, as explicited in Definition I.3.4, is the linear operator* $\widehat{\mathcal{O}}_{h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{V}}}$ *on* $\Gamma\big(\mathcal{L}_{\sigma,\mathcal{B}|(\pi,\varepsilon)}\big)$ *with restrictions*

$$
\begin{aligned}
&\widehat{\mathcal{O}}_{h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{V}}}\big|_{\mathrm{pr}^{-1}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}\left(\mathcal{O}_{(\pi,\varepsilon)\,\mathrm{i}}\right)} \qquad (5.26)\\
&= -\mathrm{i}\,\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\left(\mathrm{e}^{-\mathrm{pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\rhd\widetilde{\mathfrak{V}}_{\mathrm{i}}\right)}+\Big\langle\mathrm{e}^{-\mathrm{pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\rhd\widetilde{\mathfrak{V}}_{\mathrm{i}},\mathfrak{T}_{\mathcal{B}\,|\,(\pi,\varepsilon)}\Big\rangle=:\widehat{h}^{\mathcal{B}\,|\,\varepsilon}_{\widetilde{\mathfrak{V}}_{\mathrm{i}}}\,,
\end{aligned}
$$

expressed in terms of local sections

$$
\widetilde{\mathfrak{V}}_{\mathrm{i}}:=\mathrm{e}^{-E_{(\pi,\varepsilon)\,\mathrm{i}}}\rhd\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\in\Big(\mathsf{E}^{(1,0)}_{\mathrm{pr}^*_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}\mathcal{L}_{(\mathcal{G},\mathcal{B})|(\pi,\varepsilon)}}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\Big)\big(\mathrm{pr}^{-1}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}\big(\mathcal{O}_{(\pi,\varepsilon)\,\mathrm{i}}\big)\big)\,.\tag{5.27}
$$

Given two σ-symmetric ι_α-aligned sections $\mathfrak{V},\mathfrak{W}$, the Poisson bracket of the associated hamiltonian functions, determined by $\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ in the manner detailed in Remark I.3.3, reads

$$
\{h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{V}},h^{\mathcal{B}\,|\,\varepsilon}_{\mathfrak{W}}\}_{\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}=h^{\mathcal{B}\,|\,\varepsilon}_{[\![\,\mathfrak{V}\,,\,\mathfrak{W}\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}}\,.\tag{5.28}
$$

The commutator of the corresponding pre-quantum hamiltonians is (locally) given by

$$
\big[\widehat{h}^{\mathcal{B}\,|\,\varepsilon}_{\widetilde{\mathfrak{V}}_{\mathrm{i}}},\widehat{h}^{\mathcal{B}\,|\,\varepsilon}_{\widetilde{\mathfrak{W}}_{\mathrm{i}}}\big]=-\mathrm{i}\,\widehat{h}^{\mathcal{B}|\varepsilon}_{[\,\widetilde{\mathfrak{V}}_{\mathrm{i}}\,,\,\widetilde{\mathfrak{W}}_{\mathrm{i}}\,]_{\mathrm{V}}}\,.\tag{5.29}
$$

Proof. The proof proceeds along the same lines as for Proposition 3.12. Thus, we first rewrite the symplectic form of the 1-twisted sector from Eq. (3.4) as

$$\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}=\delta_{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}{\rm H}+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega}\mathfrak{T}_{\mathcal{B}\,|\,(\pi,\varepsilon)}\,.$$

Take an arbitrary $\mathfrak{V}\in\Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ and denote its restrictions to M and Q as $\mathfrak{V}|_M={}^M\mathscr{V}\oplus\upsilon$ and $\mathfrak{V}|_Q={}^Q\mathscr{V}\oplus\xi$, respectively, representing $\mathfrak{V}$ as $\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\oplus(\upsilon,\xi)$. Then, using conditions (5.13) and (5.14) together with Eqs. (5.12), (5.19) and (5.20), the conditions of σ-symmetricity and ι_α-alignment of $\mathfrak{V}$, and, finally, Eqs. (5.18) and (5.17), we obtain,

$$\begin{aligned}
&\alpha_{\mathsf{TP}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\big)\lrcorner\,\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\\
=\;&\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,\big(\delta{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}+\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}{\rm H}+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega\big)\\
=\;&-\delta\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\big)-\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\big({}^M\mathscr{V}\lrcorner\,{\rm H}\big)+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\big({}^Q\mathscr{V}\lrcorner\,\omega\big)\\
=\;&-\delta\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\lrcorner\,{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}+\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\upsilon\big)+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\big(\Delta_Q\upsilon+{}^Q\mathscr{V}\lrcorner\,\omega\big)\\
=\;&-\delta\Big\langle\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{\iota_\alpha\,*}\big({}^M\mathscr{V},{}^Q\mathscr{V}\big)\oplus\big(\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\upsilon+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\xi\big),\mathfrak{T}_{\mathcal{B}\,|\,(\pi,\varepsilon)}\Big\rangle\,,
\end{aligned}$$

whence Eq. (5.25) ensues upon invoking the definition of the lift $\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}$. The pre-quantum hamiltonian can then be reproduced, in the form stipulated, by specialisation of the general definition (I.3.8), and we easily see, through direct inspection, that the local objects $\widetilde{\mathfrak{V}}_{\rm i}$ are in the image of an isomorphism defined analogously to the isomorphism ${}^{\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\chi^{-1}$ from the constructive proof of Theorem 5.8.

As a corollary to the above, we obtain a hamiltonian section $\widetilde{\mathfrak{V}}\in\Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}\big)$ for every σ-symmetric ι_α-aligned section $\mathfrak{V}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$, given by

$$\widetilde{\mathfrak{V}}={\rm e}^{{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\vartriangleright\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\,.$$

Consider a pair of sections $\mathfrak{V},\mathfrak{W}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ and the respective hamiltonian sections $\widetilde{\mathfrak{V}}$ and $\widetilde{\mathfrak{W}}$. The Poisson bracket of the corresponding hamiltonian functions can be extracted from the canonical Vinogradov bracket

$$\begin{aligned}
\big[\widetilde{\mathfrak{V}},\widetilde{\mathfrak{W}}\big]^{\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}_{\rm V}\;&\equiv\;\Big[{\rm e}^{{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\vartriangleright\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\,,\,{\rm e}^{{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\vartriangleright\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{W}\Big]^{\Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}_{\rm V}\\
&=\;{\rm e}^{{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\vartriangleright\Big[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\,,\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{W}\Big]^{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}{\rm H}+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega}_{\rm V}\\
&=\;{\rm e}^{{\rm pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\vartriangleright\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\big(\llbracket\mathfrak{V}\,,\,\mathfrak{W}\rrbracket^{({\rm H},\omega;\Delta_Q)}\big)\equiv\widetilde{\llbracket\mathfrak{V}\,,\,\mathfrak{W}\rrbracket^{({\rm H},\omega;\Delta_Q)}}\,,
\end{aligned}$$

calculated with the help of the results from the proof of Proposition 2.5 and Eq. (5.24). This proves Eq. (5.28).

Upon (partially) reversing the last chain of equalities, using Eq. (I.3.9) and introducing the local data $\big(\theta_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)\,{\rm i}},\gamma_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)\,{\rm ij}}\big)$ of the pre-quantum bundle from Corollary I.3.19 (associated with the open cover $\mathcal{O}_{\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}=\{{\rm pr}^{-1}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}\big(\mathcal{O}_{(\pi,\varepsilon)\,{\rm i}}\big)\}_{{\rm i}\in\mathscr{I}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}}$), we readily derive the commutator of the (local) pre-quantum hamiltonians,

$$\begin{aligned}
\big[\widehat{h}^{\mathcal{B}|\varepsilon}_{\widetilde{\mathfrak{V}}_{\rm i}},\widehat{h}^{\mathcal{B}|\varepsilon}_{\widetilde{\mathfrak{W}}_{\rm i}}\big]\;&=\;-{\rm i}\,\widehat{\mathcal{O}}_{\{h^{\mathcal{B}|\varepsilon}_{\mathfrak{V}},h^{\mathcal{B}|\varepsilon}_{\mathfrak{W}}\}_{\Omega_{\sigma,\mathcal{B}|(\pi,\varepsilon)}}}\big\vert_{{\rm pr}^{-1}_{\mathsf{L}_{Q|(\pi,\varepsilon)}M}\left(\mathcal{O}_{(\pi,\varepsilon)\,{\rm i}}\right)}\\
&=\;-{\rm i}\Bigg(-{\rm i}\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\left({\rm e}^{-\theta_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)\,{\rm i}}}\vartriangleright\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\llbracket\mathfrak{V}\,,\,\mathfrak{W}\rrbracket^{({\rm H},\omega;\Delta_Q)}\right)}\\
&\qquad+\Big\langle{\rm e}^{-\theta_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)\,{\rm i}}}\vartriangleright\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\llbracket\mathfrak{V}\,,\,\mathfrak{W}\rrbracket^{({\rm H},\omega;\Delta_Q)},\mathfrak{T}_{\mathcal{B}\,|\,(\pi,\varepsilon)}\Big\rangle\Bigg)\\
&=\;-{\rm i}\Bigg(-{\rm i}\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}}\left({\rm e}^{-\theta_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)\,{\rm i}}}\vartriangleright\left[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\,,\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{W}\right]^{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}{\rm H}+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega}_{\rm V}\right)}\\
&\qquad+\Big\langle{\rm e}^{-\theta_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)\,{\rm i}}}\vartriangleright\Big[\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{V}\,,\,\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}\mathfrak{W}\Big]^{\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}{\rm H}+\varepsilon\,\underline{\widetilde{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\omega}_{\rm V},\mathfrak{T}_{\mathcal{B}\,|\,(\pi,\varepsilon)}\Big\rangle\Bigg)
\end{aligned}$$

$$= \quad -\mathsf{i}\Bigg(-\mathsf{i}\mathscr{L}_{\alpha_{\mathsf{TP}_{\sigma,\mathcal{B}|(\pi,\varepsilon)}}}\Big(\mathrm{e}^{-\mathrm{pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\triangleright\big[\widetilde{\mathfrak{V}}_{\mathsf{i}}\,,\widetilde{\mathfrak{W}}_{\mathsf{i}}\,\big]_{\mathrm{V}}\Big)$$
$$+\Big\langle\mathrm{e}^{-\mathrm{pr}^*_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}\theta_{\mathsf{T}^*\mathsf{L}_{Q|(\pi,\varepsilon)}M}}\triangleright\big[\widetilde{\mathfrak{V}}_{\mathsf{i}}\,,\widetilde{\mathfrak{W}}_{\mathsf{i}}\,\big]_{\mathrm{V}}\,,\mathfrak{T}_{\mathcal{B}|(\pi,\varepsilon)}\Big\rangle\Bigg)\,,$$

in conformity with Eq. (5.29). This completes the proof of the proposition. □

It is natural to ask about the conditions under which the symplectic realisation of the internal symmetries of the σ-model on the twisted sector of the theory becomes hamiltonian. A clear-cut answer is best phrased upon organising the symplectic data in hand in a manner similar to the untwisted case. Thus,

Proposition 5.10. *Adopt the notation of Definitions 2.4 and 5.1, and of Proposition 5.5. The subspace* $\alpha_{\mathsf{T}(M\sqcup Q)}\big(\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)\big)$ *is a Lie subalgebra, to be denoted as* $\mathfrak{g}_\sigma$*, within the Lie algebra of vector fields on* $M\sqcup Q$ *with a Killing restriction to* M*. Fix a basis* $\{\mathscr{K}_A\}_{A\in\overline{1,\dim\mathfrak{g}_\sigma}}$*, with restrictions* $\mathscr{K}_A|_M={}^M\mathscr{K}_A$ *and* $\mathscr{K}_A|_Q={}^Q\mathscr{K}_A$ *such that the defining commutation relations*

$$[\mathscr{K}_A,\mathscr{K}_B]=f_{ABC}\,\mathscr{K}_C$$

hold true for some structure constants f_{ABC}*. Assuming that condition 5.1 is satisfied, the corresponding* σ*-symmetric* ι_α*-aligned sections* $\mathfrak{K}_A$ *with restrictions*

$$\mathfrak{K}_A|_M={}^M\mathscr{K}_A\oplus\kappa_A=:{}^M\mathfrak{K}_A\,,\qquad\qquad \mathfrak{K}_A|_Q={}^Q\mathscr{K}_A\oplus k_A=:{}^Q\mathfrak{K}_A\,,$$

$$\begin{cases}\mathscr{L}_{{}^M\mathscr{K}_A}\mathrm{g}=0\\[4pt] \mathsf{d}_{\mathrm{H}}{}^M\mathfrak{K}_A=0\end{cases}\,,\qquad\qquad \begin{cases}\iota_{\alpha\,*}{}^Q\mathscr{K}_A={}^M\mathscr{K}_A|_{\iota_\alpha(Q)}\\[4pt] \mathsf{d}_\omega{}^Q\mathfrak{K}_A=-\Delta_Q\kappa_A\end{cases}$$

and the canonical contraction (with a trivial restriction to Q*)*

$$\mathrm{c}_{(AB)}=(\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,)_\lrcorner|_M$$

satisfy the relations

$$[\![\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}=f_{ABC}\,\mathfrak{K}_C+0\oplus\alpha_{AB}\tag{5.30}$$

with

$$\alpha_{AB}|_{\mathscr{M}}=\begin{cases}\mathscr{L}_{{}^M\mathscr{K}_A}\kappa_B-f_{ABC}\,\kappa_C-\mathsf{d}\mathrm{c}_{(AB)} & \text{on}\quad \mathscr{M}=M\\[4pt] \mathscr{L}_{{}^Q\mathscr{K}_A}k_B-f_{ABC}\,k_C+\Delta_Q\mathrm{c}_{(AB)} & \text{on}\quad \mathscr{M}=Q\end{cases}\,.$$

Proof. Obvious, through inspection. □

The symplectic realisation of the symmetries is further characterised in

Proposition 5.11. *In the notation of Theorem 4.1, of Lemma 5.7, of Theorem 5.8, and of Propositions 5.9 and 5.10, the sections* $\mathfrak{K}_A$ *determine a symplectic realisation of* $\mathfrak{g}_\sigma$ *on* $C^\infty\big(\mathsf{P}_{\sigma,\mathcal{B}|(\pi,\varepsilon)},\mathbb{R}\big)$ *by hamiltonian functions* $h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_A}$*, and an operator realisation of* $\mathfrak{g}_\sigma$ *on* $\Gamma\big(\mathcal{L}_{\sigma,\mathcal{B}|(\pi,\varepsilon)}\big)$ *by pre-quantum hamiltonians* $\widehat{\mathcal{O}}_{h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_A}}$ *with local restrictions* $\widehat{h}^{\mathcal{B}|\varepsilon}_{\widetilde{\mathfrak{K}}_{A\,\mathsf{i}}}$*. The former realisation is hamiltonian,*

$$\{h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_A},h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_B}\}_{\Omega_{\sigma,\mathcal{B}|(\pi,\varepsilon)}}=f_{ABC}\,h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_C}\,,\tag{5.31}$$

iff the $\mathfrak{K}_A$ *can be chosen such that*

$$\mathscr{L}_{{}^M\mathscr{K}_A}\kappa_B=f_{ABC}\,\kappa_C+\mathsf{d}{}^MD_{AB}\,,\qquad\qquad \mathscr{L}_{{}^Q\mathscr{K}_A}k_B=f_{ABC}\,k_C-\Delta_Q{}^MD_{AB}-{}^QD_{AB}\tag{5.32}$$

for some ${}^MD_{AB}\in C^\infty(M,\mathbb{R})$ *and (local) constants* ${}^QD_{AB}$*. In this case also*

$$[\![\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]\!]^{(\mathrm{H},\omega;\Delta_Q)}=f_{ABC}\,\mathfrak{K}_C+0\oplus\big(\mathsf{d}({}^MD_{AB}-\mathrm{c}_{(AB)}),-\Delta_Q({}^MD_{AB}-\mathrm{c}_{(AB)})-{}^QD_{AB}\big)\tag{5.33}$$

and

$$\big[\widetilde{\mathfrak{K}}_{A\,\mathsf{i}}\,,\,\widetilde{\mathfrak{K}}_{B\,\mathsf{i}}\big]_{\mathrm{V}}=f_{ABC}\,\widetilde{\mathfrak{K}}_{C\,\mathsf{i}}\,.\tag{5.34}$$

The latter identity then implies

$$[\widehat{h}^{\mathcal{B}|\varepsilon}_{\widetilde{\mathfrak{K}}_A\,\mathrm{i}},\widehat{h}^{\mathcal{B}|\varepsilon}_{\widetilde{\mathfrak{K}}_B\,\mathrm{i}}] = -\mathrm{i}\,f_{ABC}\,\widehat{h}^{\mathcal{B}|\varepsilon}_{\widetilde{\mathfrak{K}}_C\,\mathrm{i}}\,. \tag{5.35}$$

Proof. A proof is given in Section 7.2.3. It invokes some elementary facts from the theory of singular and differential (co)homology of paired manifolds $Q \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} M$ of the kind discussed in the last two sections. The relevant formalism will be set up in Section 7. □

We have established a physically motivated algebraic structure on the space of (distinguished) sections of generalised tangent bundles over the composite target space of the non-linear σ-model in the presence of circular defects in the world-sheet. The structure can be understood as a target-space model of the Poisson algebra of Noether charges of rigid symmetries of the σ-model. Prior to giving it an interpretation independent of the physical context of interest, we pause to complete the canonical description of the symmetries for a generic multi-phase σ-model, admitting the possibility of self-intersecting defects.

6. Intertwiners of the symmetry algebra from inter-bi-brane data

The physical Leitmotiv of the analysis carried out in the foregoing sections was to understand mechanisms of symmetry transmission across conformal defects, and – in this manner – to pave the way to adding more structure to the correspondence between defects and σ-model dualities worked out in Section I.4 by deriving constraints under which not merely the Virasoro modules in the state space of the (quantum) theory but also their submodules closed under the action of an extended current symmetry algebra are mapped into one another by the symplectomorphism (resp. by the endomorphism of the pre-quantum bundle) defined by the data of the defect. In the present section, we bring this line of thought to its logical conclusion and restate the questions concerning the fate of the internal symmetries at the defect quiver in the setting of Section I.5, that is for state spaces under fusion. Based on the findings of that section, it is well-justified to expect that the data carried by defect junctions of those defect quivers whose defect lines are transmissive to some internal symmetries of the untwisted sector of the theory give rise to intertwiners between representations of the symmetry algebra furnished by the state spaces under fusion. This expectation will be rendered rigorous and then proven below. For the sake of transparency of the discussion, we shall restrict it to the simplest non-trivial configurations of state spaces under fusion, to wit, those studied in Section I.5. For the same reason, we shall also extract the physically relevant structures from the extensive algebraic framework set up earlier in the paper and proceed with our reasoning in a completely explicit fashion, leaving a more abstract formulation of the results as an exercise for the interested reader.

Cross-defect fusion processes generically involve non-trivial defect junctions. Therefore, a prerequisite for our subsequent discussion is a geometric description of infinitesimal (rigid) symmetries of the σ-model on world-sheets with arbitrary embedded defect quivers, which we can infer from Proposition 4.3. Proposition 4.4, valid for circular defects, is now generalised to

Proposition 6.1. *Adopt the notation of Definitions 2.4, 2.14 and 3.2. Denote by* $\mathrm{H}\in Z^3(M)$ *the curvature of the gerbe* $\mathcal{G}$*, and write*

$$\Delta_Q := \iota_2^* - \iota_1^*\,, \qquad\qquad \Delta_{T_n} := \sum_{k=1}^n \varepsilon_n^{k,k+1}\,\pi_n^{k,k+1\,*}\,. \tag{6.1}$$

Infinitesimal rigid symmetries of the two-dimensional non-linear σ-model for network-field configurations $(X\,|\,\Gamma)$ in string background $\mathfrak{B}$ on world-sheet (Σ,γ) with a defect quiver Γ, as described in Definition I.2.7, correspond to triples $({}^M\mathfrak{V},{}^Q\mathfrak{V},{}^{T_n}\mathscr{V})$ consisting of a σ-symmetric section ${}^M\mathfrak{V}\in\Gamma_\sigma\big(\mathsf{E}^{(1,1)}M\big)$ of $\mathsf{E}^{(1,1)}M$, as defined by Eq. (4.7), of a ${}^M\mathfrak{V}$-twisted σ-symmetric section ${}^Q\mathfrak{V}\in\Gamma\big(\mathsf{E}^{(1,0)}Q\big)$ of $\mathsf{E}^{(1,0)}Q$, as defined by Eq. (4.8) and relations

$$\Delta_{T_n}\mathrm{pr}_{C^\infty(Q,\mathbb{R})}({}^Q\mathfrak{V}) = 0\,, \tag{6.2}$$

*written in terms of the canonical projection $\mathrm{pr}_{C^\infty(Q,\mathbb{R})}:\mathsf{E}^{(1,0)}Q\to C^\infty(Q,\mathbb{R})$, and of a family of vector fields ${}^{T_n}\mathscr{V}$ on the respective manifolds T_n. These are subject to the $(\iota_\alpha,\pi_n^{k,k+1})$-**alignment conditions***

$$\alpha_{\mathsf{T}M}({}^M\mathfrak{V})\big|_{\iota_\alpha(Q)} = \iota_{\alpha\,*}\alpha_{\mathsf{T}Q}({}^Q\mathfrak{V})\,, \qquad\qquad \alpha_{\mathsf{T}Q}({}^Q\mathfrak{V})\big|_{\pi_n^{k,k+1}(Q)} = \pi_{n\,*}^{k,k+1}\,{}^{T_n}\mathscr{V}\,. \tag{6.3}$$

We are now ready to study at length the issue of charge conservation at generic interaction vertices in the canonical framework developed earlier.

As the first configuration of state spaces under fusion, we treat the situation illustrated in Figure I.5, that is we consider pairs of states from the untwisted sector of the σ-model fused across the defect. The first obvious issue is the definition of the hamiltonian functions and of the corresponding pre-quantum hamiltonians for those symmetries of the untwisted sector which are transmitted across the defect, in the sense of Proposition 4.4.

Corollary 6.2. *Adopt the notation of Definitions 2.4, 3.2 and 5.1, of Propositions 3.3, 3.12 and 5.5, and of Theorem 3.11. Let* $\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}$ *be the* $\mathcal{B}$*-fusion subspace of the untwisted string from Definition I.5.4, with the choice* $\mathcal{O}_{\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}} = \{\mathcal{O}_{(\mathrm{i}^1,\mathrm{i}^2)}^{\otimes\mathcal{B}}\}_{\mathrm{i}^1,\mathrm{i}^2\in\mathscr{I}_{\mathsf{L}M}}$ *of an open cover induced from the (sufficiently fine) open cover* $\mathcal{O}_{\mathsf{L}M}$ *in the manner detailed in the proof of Theorem I.5.5. Take an arbitrary* σ*-symmetric* ι_α*-aligned section* $\mathfrak{V}\in\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ *with restrictions* $\mathfrak{V}|_M = {}^M\mathscr{V}\oplus\upsilon$ *and* $\mathfrak{V}|_Q = {}^Q\mathscr{V}\oplus\xi$*. Write* $\mathsf{I}=[0,\pi]$ *and let* $\tau:\mathbb{S}^1\to\mathbb{S}^1$ *be the* π*-shift map from Eq. (I.5.2). The hamiltonian function* $h_{\mathfrak{V}}^{\otimes\mathcal{B}}$ *on* $\mathsf{P}_{\sigma,\varnothing}^{\times 2}$ *associated to* $\mathfrak{V}$ *restricts to* $\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}$ *as*

$$\begin{aligned} h_{\mathfrak{V}}^{\otimes\mathcal{B}}[(\psi_1,\psi_2)] &= \int_{\mathsf{I}}\mathrm{Vol}(\mathsf{I})\left[{}^M\mathscr{V}\big(X_2(\cdot)\big)\lrcorner\,\mathsf{p}_2+(X_{2\,*}\widehat{t})\lrcorner\,\upsilon\big(X_2(\cdot)\big)\right] \\ &\quad+\int_{\tau(\mathsf{I})}\mathrm{Vol}\big(\tau(\mathsf{I})\big)\left[{}^M\mathscr{V}\big(X_1(\cdot)\big)\lrcorner\,\mathsf{p}_1+(X_{1\,*}\widehat{t'})\lrcorner\,\upsilon\big(X_1(\cdot)\big)\right]+Y_{1,2}^*\xi(\pi)-Y_{1,2}^*\xi(0)\,, \end{aligned}\tag{6.4}$$

written in terms of the tangent vector field $\widehat{t}$ *and the volume form* $\mathrm{Vol}(\mathsf{I})$ *on* I*, the tangent vector field* $\widehat{t'}$ *and the volume form* $\mathrm{Vol}\big(\tau(\mathsf{I})\big)$ *on* $\tau(\mathsf{I})$*, and for an arbitrary pair* $(\psi_1,\psi_2)\in\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}$ *of states, represented by the respective Cauchy data* $\psi_\alpha=(X_\alpha,\mathsf{p}_\alpha)$*,* $\alpha\in\{1,2\}$ *and glued along the open path* $Y_{1,2}\in\mathsf{I}Q$*. The corresponding pre-quantum hamiltonian for* $h_{\mathfrak{V}}^{\otimes\mathcal{B}}$*, constructed in conformity with Definition I.3.4, has local restrictions*

$$\widehat{\mathcal{O}}_{h_{\mathfrak{V}}^{\otimes\mathcal{B}}}|_{\mathcal{O}_{(\mathrm{i}^1,\mathrm{i}^2)}^{\otimes\mathcal{B}}} = -\mathsf{i}\mathscr{L}_{({}^M\widetilde{\mathscr{V}},{}^Q\widetilde{\mathscr{V}})}-\big({}^M\widetilde{\mathscr{V}},{}^Q\widetilde{\mathscr{V}}\big)\lrcorner\,\theta_{\sigma,\otimes\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)}+h_{\mathfrak{V}}^{\otimes\mathcal{B}}=:\widehat{h}_{\mathfrak{V}\,(\mathrm{i}^1,\mathrm{i}^2)}^{\otimes\mathcal{B}}\,,\tag{6.5}$$

expressed in terms of the restrictions

$$\begin{aligned} {}^M\widetilde{\mathscr{V}}[(\psi_1,\psi_2)] &= \int_{\mathsf{I}}\mathrm{Vol}(\mathsf{I})\left[{}^M\mathscr{V}^\mu\big(X_2(\cdot)\big)\tfrac{\delta}{\delta X_2^\mu(\cdot)}-\mathsf{p}_{2\,\mu}(\cdot)\,\partial_\nu{}^M\mathscr{V}^\mu\big(X_2(\cdot)\big)\tfrac{\delta}{\delta\mathsf{p}_{2\,\nu}(\cdot)}\right] \\ &\quad+\int_{\tau(\mathsf{I})}\mathrm{Vol}\big(\tau(\mathsf{I})\big)\left[{}^M\mathscr{V}^\mu\big(X_1(\cdot)\big)\tfrac{\delta}{\delta X_1^\mu(\cdot)}-\mathsf{p}_{1\,\mu}(\cdot)\,\partial_\nu{}^M\mathscr{V}^\mu\big(X_1(\cdot)\big)\tfrac{\delta}{\delta\mathsf{p}_{1\,\nu}(\cdot)}\right] \\ {}^Q\widetilde{\mathscr{V}}[(\psi_1,\psi_2)] &= \int_{\mathsf{I}}\mathrm{Vol}(\mathsf{I})\,{}^Q\mathscr{V}^A\big(Y_{1,2}(\cdot)\big)\tfrac{\delta}{\delta Y_{1,2}^A(\cdot)} \end{aligned}$$

of the lift of the vector-field component of $\mathfrak{V}$ *to* $\mathsf{P}_{\sigma,\varnothing}^{\times 2}$*, and of the local data* $\big(\theta_{\sigma,\otimes\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)},\gamma_{\sigma,\otimes\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)(\mathrm{j}^1,\mathrm{j}^2)}\big)$*, derived in the proof of Theorem I.5.5, of the restriction* $\mathcal{L}_{\sigma,\otimes\mathcal{B}}=\big(\mathrm{pr}_1^*\mathcal{L}_{\sigma,\varnothing}\otimes\mathrm{pr}_2^*\mathcal{L}_{\sigma,\varnothing}\big)|_{\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}}$ *to* $\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}$ *of the tensor product of pullbacks of* $\mathcal{L}_{\sigma,\varnothing}$ *along the canonical projections* $\mathrm{pr}_\alpha:\mathsf{P}_{\sigma,\varnothing}^{\times 2}\to\mathsf{P}_{\sigma,\varnothing}$*.*

Proof. The formula for $h_{\mathfrak{V}}$ readily follows from the expression for the restricted symplectic form $\overline{\Omega}_{\sigma,\varnothing}^+$, given in Eq. (I.D.1) and taken in conjunction with the condition of the σ-symmetricity of $\mathfrak{V}$, whereas Eq. (6.5) is a specialisation of the general definition (I.3.8). □

The last corollary forms the basis of the following important result:

Theorem 6.3. *Adopt the notation of Definitions 2.4 and 3.2, of Propositions 3.3, 3.12 and 5.5, and of Corollary 6.2. Let* $\mathfrak{I}_\sigma(\otimes\mathcal{B}:\mathcal{J}:\mathcal{B})$ *be the* $2\to 1$ *cross-*$(\mathcal{B},\mathcal{J})$ *interaction subspace of the untwisted string within* $\mathsf{P}_{\sigma,\varnothing}^{\times 3}=\mathsf{P}_{\sigma,\varnothing}\times\mathsf{P}_{\sigma,\varnothing}\times\mathsf{P}_{\sigma,\varnothing}$ *described in that definition. Finally, let* $\mathfrak{V}$ *be a* σ*-symmetric section from* $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ *to which there are associated hamiltonian functions:* $h_{\mathfrak{V}}$ *on* $\mathsf{P}_{\sigma,\varnothing}$ *and* $h_{\mathfrak{V}}^{\otimes\mathcal{B}}$ *on* $\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}$*. The values attained by the pullbacks* $\mathrm{pr}_3^*h_{\mathfrak{V}}$ *and* $(\mathrm{pr}_1\times\mathrm{pr}_2)^*h_{\mathfrak{V}}^{\otimes\mathcal{B}}$ *along the canonical projections* $\mathrm{pr}_n:\mathsf{P}_{\sigma,\varnothing}^{\times 3}\to\mathsf{P}_{\sigma,\varnothing}$*,* $n\in\{1,2,3\}$ *coincide on* $\mathfrak{I}_\sigma(\otimes\mathcal{B}:\mathcal{J}:\mathcal{B})$ *iff the condition*

$$\big(\pi_3^{1,2\,*}+\pi_3^{2,3\,*}-\pi_3^{3,1\,*}\big)\mathrm{pr}_{C^\infty(Q,\mathbb{R})}(\mathfrak{V})=0\tag{6.6}$$

is satisfied on T_3 *for the canonical projection* $\mathrm{pr}_{C^\infty(Q,\mathbb{R})}:\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\to C^\infty(Q,\mathbb{R})$*. Furthermore, assuming that* $\mathfrak{I}_\sigma(\otimes\mathcal{B}:\mathcal{J}:\mathcal{B})$ *projects (canonically) onto each of the three cartesian factors in* $\mathsf{P}_{\sigma,\varnothing}^{\times 3}$*, the unitary similarity transformation between the set of pre-quantum hamiltonians on* $\mathsf{P}_{\sigma,\varnothing}^{\otimes\mathcal{B}}$ *and those on*

$\mathsf{P}_{\sigma,\varnothing}$ *defined by the bundle isomorphism* $\mathfrak{J}_{\sigma,(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})}$ *from Theorem I.5.5 preserves (element-wise) the respective subalgebras composed of those pre-quantum hamiltonians which are assigned, in the manner explicited in Proposition 3.12 and Corollary 6.2, respectively, to the* ι_α*-aligned* σ*-symmetric sections of* $\mathsf{E}^{(1,1)}M \sqcup \mathsf{E}^{(1,0)}Q$ *iff the same condition holds true.*

Proof. The proof goes along the same lines as those of Propositions 4.5 and 4.6. Take a ι_α-aligned σ-symmetric section $\mathfrak{V}$ with restrictions $\mathfrak{V}|_M = {}^M\mathscr{V} \oplus \upsilon$ and $\mathfrak{V}|_Q = {}^Q\mathscr{V} \oplus \xi$. In the classical setting, we compute, substituting the defining relations (I.5.9)-(I.5.11) of $\mathfrak{J}_\sigma(\circledast\mathcal{B} : \mathcal{J} : \mathcal{B}) \ni (\psi_1, \psi_2, \psi_3)$, $\psi_n = (X_n, \mathsf{p}_n)$, $n \in \{1,2,3\}$ in Eq. (6.4),

$$\begin{aligned}
h^{\circledast\mathcal{B}}_{\mathfrak{V}}[(\psi_1,\psi_2)] &= \int_{\mathsf{I}} \mathrm{Vol}(\mathsf{I}) \left[{}^Q\mathscr{V}\big(Y_{2,3}(\cdot)\big) \lrcorner (\mathsf{p}_2 \circ \iota_{1\,*}) + (Y_{2,3\,*}\widehat{t}) \lrcorner \iota_1^*\upsilon\big(Y_{2,3}(\cdot)\big)\right] \\
&\quad + \int_{\tau(\mathsf{I})} \mathrm{Vol}\big(\tau(\mathsf{I})\big) \left[{}^Q\mathscr{V}\big(Y_{1,3}(\cdot)\big) \lrcorner (\mathsf{p}_1 \circ \iota_{1\,*}) + (Y_{1,3\,*}\widehat{t}') \lrcorner \iota_1^*\upsilon\big(Y_{1,3}(\cdot)\big)\right] + Y_{1,2}^*\xi(\pi) - Y_{1,2}^*\xi(0) \\
&= \int_{\mathsf{I}} \mathrm{Vol}(\mathsf{I}) \left[{}^Q\mathscr{V}\big(Y_{2,3}(\cdot)\big) \lrcorner (\mathsf{p}_3 \circ \iota_{2\,*}) + (Y_{2,3\,*}\widehat{t}) \lrcorner \big(\iota_1^*\upsilon - {}^Q\mathscr{V} \lrcorner \omega\big)\big(Y_{2,3}(\cdot)\big)\right] \\
&\quad + \int_{\tau(\mathsf{I})} \mathrm{Vol}\big(\tau(\mathsf{I})\big) \left[{}^Q\mathscr{V}\big(Y_{1,3}(\cdot)\big) \lrcorner (\mathsf{p}_3 \circ \iota_{2\,*}) + (Y_{1,3\,*}\widehat{t}') \lrcorner \big(\iota_2^*\upsilon - {}^Q\mathscr{V} \lrcorner \omega\big)\big(Y_{1,3}(\cdot)\big)\right] \\
&\quad + Y_{1,2}^*\xi(\pi) - Y_{1,2}^*\xi(0) \\
&= h_{\mathfrak{V}}[\psi_3] + Y_{2,3}^*\xi(\pi) - Y_{2,3}^*\xi(0) + Y_{1,3}^*\xi(0) - Y_{1,3}^*\xi(\pi) + Y_{1,2}^*\xi(\pi) - Y_{1,2}^*\xi(0) \\
&= h_{\mathfrak{V}}[\psi_3] + Z^*\big(\pi_3^{1,2\,*} + \pi_3^{2,3\,*} - \pi_3^{3,1\,*}\big)\big(\xi(\pi) - \xi(0)\big)\,,
\end{aligned}$$

whence the first statement of the theorem follows.

Passing to the pre-quantum régime, fix an open cover $\mathcal{O}_{\mathfrak{J}_\sigma(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})} = \{\mathcal{O}^*_{\mathrm{i}^1} \times \mathcal{O}^*_{\mathrm{i}^2} \times \mathcal{O}^*_{\mathrm{i}^3}\}_{\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3 \in \mathscr{I}_{\mathsf{L}M}}$ as in the proof of Theorem I.5.5 and take the associated data $\mathrm{pr}_n^*(\theta_{\sigma,\varnothing\,\mathrm{i}^n}, \gamma_{\sigma,\varnothing\,\mathrm{i}^n\mathrm{j}^n})$, $n \in \{1,2,3\}$ of the pullback bundles $\mathrm{pr}_n^*\mathcal{L}_{\sigma,\varnothing}$, and those of the bundle isomorphism $\mathfrak{J}_{\sigma,(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})}$, denoted by $f^{+-}_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3)}$ and given in Eq. (I.D.4). The latter determine a relation between local sections $s_{(\mathrm{i}^1,\mathrm{i}^2)} \coloneqq \mathrm{pr}_1^*s_{\mathrm{i}^1} \otimes \mathrm{pr}_2^*s_{\mathrm{i}^2} : \mathcal{O}^*_{\mathrm{i}^1} \times \mathcal{O}^*_{\mathrm{i}^2} \times \mathcal{O}^*_{\mathrm{i}^3} \to \mathrm{pr}_1^*\mathcal{L}_{\sigma,\varnothing} \otimes \mathrm{pr}_2^*\mathcal{L}_{\sigma,\varnothing}$ and $\mathrm{pr}_3^*s_{\mathrm{i}^3} : \mathcal{O}^*_{\mathrm{i}^1} \times \mathcal{O}^*_{\mathrm{i}^2} \times \mathcal{O}^*_{\mathrm{i}^3} \to \mathrm{pr}_3^*\mathcal{L}_{\sigma,\varnothing}$ over $\mathfrak{J}_\sigma(\circledast\mathcal{B} : \mathcal{J} : \mathcal{B})$ of the form

$$s_{(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] = f^{+-}_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3)}[(\psi_1,\psi_2,\psi_3)] \cdot s_{\mathrm{i}^3}[\psi_3]\,.$$

Taking into account Eqs. (6.5) and (I.D.3) and using the explicit formula (3.17) for $\widehat{h}_{\mathfrak{V}_{\mathrm{i}^3}}[\psi_3]$, we obtain

$$\begin{aligned}
&\widehat{h}^{\circledast\mathcal{B}}_{\mathfrak{V}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \triangleright s_{(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \\
={}& \Big(-\mathsf{i}\mathscr{L}_{({}^M\widetilde{\mathscr{V}},{}^Q\widetilde{\mathscr{V}})[(\psi_1,\psi_2)]}|_{\psi_3=\mathrm{const}} - \mathsf{i}\mathscr{L}_{\widetilde{\mathsf{L}}_*{}^M\mathscr{V}[\psi_3]}|_{\psi_1,\psi_2=\mathrm{const}} - \big({}^M\widetilde{\mathscr{V}}, {}^Q\widetilde{\mathscr{V}}\big) \lrcorner \theta_{\sigma,\circledast\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \\
&+ h^{\circledast\mathcal{B}}_{\mathfrak{V}}[(\psi_1,\psi_2)]\Big) f^{+-}_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3)}[(\psi_1,\psi_2,\psi_3)] \cdot s_{\mathrm{i}^3}[\psi_3] \\
={}& f^{+-}_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3)}[(\psi_1,\psi_2,\psi_3)] \cdot \Big[-\mathsf{i}\mathscr{L}_{\widetilde{\mathsf{L}}_*{}^M\mathscr{V}[\psi_3]}|_{\psi_1,\psi_2=\mathrm{const}} - \big({}^M\widetilde{\mathscr{V}}, {}^Q\widetilde{\mathscr{V}}\big) \lrcorner \theta_{\sigma,\circledast\mathcal{B}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \\
&- \Big(\big({}^M\widetilde{\mathscr{V}}, {}^Q\widetilde{\mathscr{V}}\big)[(\psi_1,\psi_2)]|_{\psi_3=\mathrm{const}} + \widetilde{\mathsf{L}}_*{}^M\mathscr{V}[\psi_3]|_{\psi_1,\psi_2=\mathrm{const}}\Big) \lrcorner \mathsf{i}\,\delta\log f^{+-}_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3)}[(\psi_1,\psi_2,\psi_3)] \\
&+ h^{\circledast\mathcal{B}}_{\mathfrak{V}}[(\psi_1,\psi_2)]\Big] s_{\mathrm{i}^3}[\psi_3] \\
={}& f^{+-}_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3)}[(\psi_1,\psi_2,\psi_3)] \cdot \Big(-\mathsf{i}\mathscr{L}_{\widetilde{\mathsf{L}}_*{}^M\mathscr{V}[\psi_3]}|_{\psi_1,\psi_2=\mathrm{const}} - \widetilde{\mathsf{L}}_*{}^M\mathscr{V} \lrcorner \theta_{\sigma,\varnothing\,\mathrm{i}^3}[\psi_3] + h_{\mathfrak{V}}[\psi_3] \\
&+ Z^*\big(\pi_3^{1,2\,*} + \pi_3^{2,3\,*} - \pi_3^{3,1\,*}\big)\big(\xi(\pi) - \xi(0)\big)\Big) s_{\mathrm{i}^3}[\psi_3]
\end{aligned}$$

The above simply restates, in the setting in hand, the general rule: in the presence of an isomorphism of pre-quantum bundles, the only obstruction to having pre-quantum hamiltonians preserved by a similarity transformation induced from the isomorphism can come from non-equality of the corresponding hamiltonian functions pulled back to the graph of the underlying symplectomorphism. Thus, upon imposing Eq. (6.6), and in that case only, we find

$$\widehat{h}^{\circledast\mathcal{B}}_{\mathfrak{V}\,(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] \triangleright s_{(\mathrm{i}^1,\mathrm{i}^2)}[(\psi_1,\psi_2)] = f^{+-}_{\sigma\,(\mathrm{i}^1,\mathrm{i}^2,\mathrm{i}^3)}[(\psi_1,\psi_2,\psi_3)] \cdot \big(\widehat{h}_{\mathfrak{V}_{\mathrm{i}^3}}[\psi_3] \triangleright s_{\mathrm{i}^3}[\psi_3]\big)\,,$$

as claimed. □

Remark 6.4. The last result is the first rigorous statement concerning the anticipated relation between the geometric data carried by defect junctions of a defect quiver with defect lines that are transmissive to some internal symmetries of the untwisted sector of the σ-model and intertwiners of the algebra of those symmetries realised on the multi-string state space. It is also easily generalised to more complex interaction schemes for untwisted states – in particular, in the case of an n-string analogon of the process considered, in which untwisted states pass through a defect junction of valence n the condition of equality of the hamiltonian functions associated to a ι_α-aligned σ-symmetric section $\mathfrak{V}$ from the above proof (resp. of similarity of the corresponding pre-quantum hamiltonians) for the incoming and outgoing states takes the form

$$\Delta_{T_n} \mathrm{pr}_{C^\infty(Q,\mathbb{R})}(\mathfrak{V}) = 0 \,. \tag{6.7}$$

Comparing the latter with the characterisation of σ-model symmetries in the presence of self-intersecting defects, given in Proposition 6.1, we conclude that the charges of a σ-model symmetry that is preserved in the presence of a defect quiver are automatically additively conserved in the processes of a cross-defect splitting-joining interaction.

The conclusive piece of evidence in favour of the interpretation of the inter-bi-brane data and the the associated DJI for transmissive defects in terms of intertwiners of the algebra of symmetries of the σ-model comes from the twisted sector, in which we consider (for the sake of concreteness) the simple fusion pattern depicted in Figure I.7, in which two 1-twisted states are fused, whereupon a single 1-twisted state is produced. We have

Theorem 6.5. *Adopt the notation of Definitions 2.4 and 3.2, and of Propositions 3.3, 5.5 and 5.9. Let* $\mathsf{P}^{\circledast\mathcal{B}_{\rm triv}}_{\sigma,\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2)}$ *be the* $\mathcal{B}_{\rm triv}$*-fusion subspace of the 1-twisted string from Definition I.5.7, and let* $\mathfrak{I}_\sigma(\circledast\mathcal{B}_{\rm triv} : \mathcal{J} : \mathcal{B}_{\rm triv})^{\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2,\varepsilon_3)}$ *be the* $2\to 1$ *cross-*$(\mathcal{B}_{\rm triv},\mathcal{J})$ *interaction subspace of the 1-twisted string within* $\mathsf{P}^{+-}_{\sigma,\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2,\varepsilon_3)} = \mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon_1)}\times\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon_2)}\times\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon_2)}$ *described in that definition. Finally, let* $\mathfrak{V}$ *be a* σ*-symmetric section from* $\Gamma_{\iota_\alpha,\sigma}\big(\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q\big)$ *with restrictions* $\mathfrak{V}|_M = {}^M\mathscr{V}\oplus\upsilon$ *and* $\mathfrak{V}|_Q = {}^Q\mathscr{V}\oplus\xi$ *to which there are associated hamiltonian functions:* $h_{\mathfrak{V}}^{\mathcal{B}\,|\,\varepsilon_3}$ *on* $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon_3)}$ *and*

$$\begin{aligned} h_{\mathfrak{V}}^{\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2)}[(\psi_1,\psi_2)] \quad &= \quad \int_{\mathsf{I}} \mathrm{Vol}(\mathsf{I})\,\big[{}^M\mathscr{V}\big(X_2(\cdot)\big)\lrcorner\,\mathsf{p}_2 + (X_{2\,*}\widehat{t})\lrcorner\,\upsilon\big(X_2(\cdot)\big)\big] \\ &\quad + \int_{\tau(\mathsf{I})} \mathrm{Vol}\big(\tau(\mathsf{I})\big)\,\big[{}^M\mathscr{V}\big(X_1(\cdot)\big)\lrcorner\,\mathsf{p}_1 + (X_{1\,*}\widehat{t'})\lrcorner\,\upsilon\big(X_1(\cdot)\big)\big] + \varepsilon_1\,\xi(q_1) + \varepsilon_2\,\xi(q_2)\,, \end{aligned}$$

on $\mathsf{P}^{\circledast\mathcal{B}_{\rm triv}}_{\sigma,\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2)} \ni (\psi_1,\psi_2)$, $\psi_\alpha = (X_\alpha,\mathsf{p}_\alpha,q_\alpha,V_\alpha)$, $\alpha\in\{1,2\}$*, the latter being written in terms of the tangent vector field* $\widehat{t}$ *and the volume form* $\mathrm{Vol}(\mathsf{I})$ *on* I*, as well as the tangent vector field* $\widehat{t'}$ *and the volume form* $\mathrm{Vol}\big(\tau(\mathsf{I})\big)$ *on* $\tau(\mathsf{I})$*. The values attained by the pullbacks* $\mathrm{pr}_3^* h_{\mathfrak{V}}^{\mathcal{B}\,|\,\varepsilon_3}$ *and* $(\mathrm{pr}_1,\mathrm{pr}_2)^* h_{\mathfrak{V}}^{\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2)}$ *along the canonical projections* $\mathrm{pr}_3 : \mathsf{P}^{+-}_{\sigma,\mathcal{B}|(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \to \mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon_3)}$ *and* $(\mathrm{pr}_1,\mathrm{pr}_2) : \mathsf{P}^{+-}_{\sigma,\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \to \mathsf{P}^{\circledast\mathcal{B}_{\rm triv}}_{\sigma,\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2)}$ *coincide on* $\mathfrak{I}_\sigma(\circledast\mathcal{B}_{\rm triv} : \mathcal{J} : \mathcal{B}_{\rm triv})^{\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2,\varepsilon_3)}$ *iff condition* (6.6) *is satisfied on* T_3*. Furthermore, assuming that* $\mathfrak{I}_\sigma(\circledast\mathcal{B} : \mathcal{J} : \mathcal{B})$ *projects (canonically) onto each of the three cartesian factors* $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon_n)}$ *in* $\mathsf{P}^{+-}_{\sigma,\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2,\varepsilon_3)}$*, the unitary similarity transformation between the set of pre-quantum hamiltonians on* $\mathsf{P}^{\circledast\mathcal{B}_{\rm triv}}_{\sigma,\mathcal{B}\,|\,(\varepsilon_1,\varepsilon_2)}$ *and those on* $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon_3)}$ *defined by the bundle isomorphism* $\mathfrak{J}^{\mathcal{B}|(\varepsilon_1,\varepsilon_2,\varepsilon_3)}_{\sigma,(\circledast\mathcal{B}_{\rm triv}:\mathcal{J}:\mathcal{B}_{\rm triv})}$ *from Theorem I.5.8 preserves (element-wise) the respective subalgebras composed of those pre-quantum hamiltonians which are assigned, in the canonical manner, to the* ι_α*-aligned* σ*-symmetric sections of* $\mathsf{E}^{(1,1)}M\sqcup\mathsf{E}^{(1,0)}Q$ *iff the same condition holds true.*

Proof. The proof is a straightforward variation of the proof of Theorem 6.3. □

Example 6.6. Symmetry transmission across the maximally symmetric WZW defects

In order to prepare the ground for subsequent analysis of the maximally symmetric WZW defects, described in Ref. [RS12] (*cf.* also Example I.2.13 for the notation used), let us first note that the sections of the generalised tangent bundle $\mathsf{E}^{(1,1)}\mathscr{G}$ over the group manifold of a Lie group $\mathscr{G}$ which define the isometries of the Cartan–Killing metric and preserve the Cartan 3-form are given by

$$\mathfrak{L}_A = L_A \oplus \big(-\tfrac{\mathsf{k}}{8\pi}\,\theta_L^A\big)\,, \qquad\qquad \mathfrak{R}_A = R_A \oplus \tfrac{\mathsf{k}}{8\pi}\,\theta_R^A$$

in terms of the components θ_L^A (resp. θ_R^A) of the left-invariant (resp. right-invariant) Maurer–Cartan 1-form $\theta_L = \theta_L^A\otimes t_A$ (resp. $\theta_R = \theta_R^A\otimes t_A$) and of the standard left-invariant (resp. right-invariant) vector

fields L_A (resp. R_A) dual to them. Here, the t_A are the generators of the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$ obeying the structure relations

$$[t_A, t_B] = \mathrm{f}_{ABC}\, t_C\,,$$

with $\mathrm{f}_{ABC} \in \mathbb{C}$ the structure constants of $\mathfrak{g}$. The sections satisfy the simple H_{k}-twisted Vinogradov-bracket algebra

$$[\,\mathfrak{L}_A\,,\,\mathfrak{L}_B\,]^{\mathrm{H}_{\mathsf{k}}}_{\mathrm{V}} = \mathrm{f}_{ABC}\,\mathfrak{L}_C\,,\qquad [\,\mathfrak{R}_A\,,\,\mathfrak{R}_B\,]^{\mathrm{H}_{\mathsf{k}}}_{\mathrm{V}} = \mathrm{f}_{ABC}\,\mathfrak{R}_C\,,\qquad [\,\mathfrak{L}_A\,,\,\mathfrak{R}_B\,]^{\mathrm{H}_{\mathsf{k}}}_{\mathrm{V}} = 0\,.$$

They generate the right and left regular translations on the group, and so yield, through definition Eq. (3.26), the right and left Kač–Moody currents $J_H = J_H^A \otimes t_A,\ H \in \{L,R\}$, respectively,

$$J_{\mathfrak{L}_A} = -\tfrac{1}{4}\,J_R^A\,,\qquad J_{\mathfrak{R}_A} = -\tfrac{1}{4}\,J_L^A\,.$$

In virtue of Proposition 3.13, the associated hamiltonian functions and pre-quantum hamiltonians furnish two independent representations of the Lie algebra $\mathfrak{g}$ of the Lie group $\mathscr{G}$. Having thus made contact with our previous considerations from Example I.2.13, we may now discuss the reduction of the bulk symmetry in the presence of the defects.

Symmetries preserved by the boundary $\mathcal{G}_{\mathsf{k}}$-bi-brane. We begin with the boundary defect and the attendant bi-brane $\mathcal{B}^\partial_{\mathsf{k}}$, for which the analysis simplifies enormously: the tangent space to a conjugacy class $\mathscr{C}_\lambda \subset Q^\partial_{\mathsf{k}}$ is spanned by the axial combinations $R_A - L_A$ of the basic right- and left-invariant vector fields on the group, and so we should look for ι_λ-aligned σ-symmetric sections of $\mathsf{E}^{(1,1)}\mathscr{G} \sqcup \mathsf{E}^{(1,0)}Q^\partial_{\mathsf{k}}$ amidst those obtained from the corresponding combinations

$$\mathfrak{R}_A - \mathfrak{L}_A = (R_A - L_A) \oplus \tfrac{\mathsf{k}}{8\pi}\,(\theta_R^A + \theta_L^A)$$

in the bulk. The latter are readily checked to satisfy the σ-symmetricity condition in the form

$$(R_A - L_A) \lrcorner\, \omega^\partial_{\mathsf{k},\lambda} - \tfrac{\mathsf{k}}{8\pi}\,\iota_\lambda^*(\theta_L^A + \theta_R^A) = 0\,.$$

Thus, we obtain a basis of ι_λ-aligned σ-symmetric sections

$$\mathfrak{A}_A = (R_A - L_A, R_A - L_A) \oplus \tfrac{\mathsf{k}}{8\pi}\,(\theta_R^A + \theta_L^A, 0)\,,$$

with the $(\mathrm{H}_{\mathsf{k}}, \omega^\partial_{\mathsf{k}}; -\iota^*_{Q^\partial_{\mathsf{k}}})$-twisted brackets

$$[\![\,\mathfrak{A}_A\,,\,\mathfrak{A}_B\,]\!]^{(\mathrm{H}_{\mathsf{k}}, \omega^\partial_{\mathsf{k}}; -\iota^*_{Q^\partial_{\mathsf{k}}})} = \mathrm{f}_{ABC}\,\mathfrak{A}_C\,.$$

We conclude that the symmetry preserved by the boundary maximally symmetric WZW defect is the adjoint (axial) component of the left-right symmetry of the defect-free theory, generated by the currents

$$J_{\mathfrak{L}_A} - J_{\mathfrak{R}_A} = \tfrac{1}{4}\,(J_L^A - J_R^A)\,,$$

and that the hamiltonian functions and pre-quantum hamiltonians assigned to the sections $\mathfrak{A}_A$ furnish a representation of a single copy of $\mathfrak{g}$.

Symmetries preserved by the non-boundary $\mathcal{G}_{\mathsf{k}}$-bi-brane. In the non-boundary case, the geometry of the world-volume Q_{k} of the $\mathcal{G}_{\mathsf{k}}$-bi-brane $\mathcal{B}_{\mathsf{k}}$, in conjunction with the choice of the maps ι_α detailed in Example I.2.13, offer – via the tangent maps $\iota_{\alpha\,*}$ – an unrestrained choice of linear combinations of the basic left- and right-invariant vector fields on the target space. Indeed, one easily verifies that the vector fields ${}^{Q_{\mathsf{k}}}L_A$ and ${}^{Q_{\mathsf{k}}}R_A$ with values

$${}^{Q_{\mathsf{k}}}L_A(g,h) = L_A(g) + (L_A - R_A)(h)\,,\qquad {}^{Q_{\mathsf{k}}}R_A(g,h) = R_A(g)$$

on the bi-brane world-volume $Q_{\mathsf{k}} \ni (g,h)$ push forward to the vector fields L_A and R_A, respectively,

$$\iota_{\alpha\,*}{}^{Q_{\mathsf{k}}}L_A = L_A\,,\qquad \iota_{\alpha\,*}{}^{Q_{\mathsf{k}}}R_A = R_A\,.$$

At this stage, it remains to calculate

$${}^{Q_{\mathsf{k}}}L_A \lrcorner\, \omega_{\mathsf{k}} + \Delta_{Q_{\mathsf{k}}}\big(-\tfrac{\mathsf{k}}{8\pi}\,\theta_L^A\big) = 0\,,\qquad {}^{Q_{\mathsf{k}}}R_A \lrcorner\, \omega_{\mathsf{k}} + \Delta_{Q_{\mathsf{k}}}\big(\tfrac{\mathsf{k}}{8\pi}\,\theta_R^A\big) = 0$$

over Q_{k}, whereupon a basis can be chosen in $\Gamma_{\iota_A,\sigma}\big(\mathsf{E}^{(1,1)}\mathscr{G} \sqcup \mathsf{E}^{(1,0)}Q_{\mathsf{k}}\big)$ with elements

$$\mathfrak{L}_A = \big(L_A, {}^{Q_{\mathsf{k}}}L_A\big) \oplus \big(-\tfrac{\mathsf{k}}{8\pi}\,\theta_L^A, 0\big)\,,\qquad \mathfrak{R}_A = \big(R_A, {}^{Q_{\mathsf{k}}}R_A\big) \oplus \big(\tfrac{\mathsf{k}}{8\pi}\,\theta_R^A, 0\big)\,.$$

In this basis, we find the $(\mathrm{H}_{\mathsf{k}}, \omega_{\mathsf{k}}; \Delta_{Q_{\mathsf{k}}})$-twisted brackets

$$[\![\,\mathfrak{L}_A\,,\,\mathfrak{L}_B\,]\!]^{(\mathrm{H}_{\mathsf{k}}, \omega_{\mathsf{k}}; \Delta_{Q_{\mathsf{k}}})} = \mathrm{f}_{ABC}\,\mathfrak{L}_C\,,\qquad [\![\,\mathfrak{R}_A\,,\,\mathfrak{R}_B\,]\!]^{(\mathrm{H}_{\mathsf{k}}, \omega_{\mathsf{k}}; \Delta_{Q_{\mathsf{k}}})} = \mathrm{f}_{ABC}\,\mathfrak{R}_C\,,$$

$$[\![\,\mathfrak{L}_A\,,\,\mathfrak{R}_B\,]\!]^{(\mathrm{H}_{\mathsf{k}}, \omega_{\mathsf{k}}; \Delta_{Q_{\mathsf{k}}})} = 0\,.$$

We are thus led to conclude that the full left-right symmetry of the defect-free theory is preserved by the defect. As the 0-form components of the ι_α-aligned σ-symmetric sections are trivial, Eq. (6.7) is satisfied, and so we have a non-anomalous realisation of the symmetry algebra on multi-string state spaces.

Remark 6.7. There is an important conclusion that can be drawn from our presentation of the symmetries preserved by the non-boundary $\mathcal{G}_\mathsf{k}$-bi-brane, to wit, it transpires that whatever the world-volume of the corresponding $(\mathcal{G}_\mathsf{k}, \mathcal{B}_\mathsf{k})$-inter-bi-brane, charges of the full $\mathfrak{g} \oplus \mathfrak{g}$-symmetry are going to be additively conserved in arbitrary cross-defect interaction processes. In the light of the world-sheet interpretation of such processes, as illustrated, *e.g.*, in Figure I.7, this observation points to the existence of a straightforward correspondence between junctions of the maximally symmetric WZW defects and spaces of intertwiners of the action of the symmetry group $\mathscr{G}$ of the (bulk) WZW model. This seems to fit nicely with the classificatory results of Ref. [FFRS07], where, in particular, the defect junctions (of valence n) in the *quantised* WZW model were related to the so-called conformal blocks for the (n-)punctured Riemann sphere. The remarkable consistency between these results, derived in the rigorous categorial quantisation scheme for the WZW σ-model, and our conclusions, based on the canonical analysis conveyed entirely in geometric terms, hinges on the identification, detailed in Ref. [Gaw99, Sec. 5], between the said conformal blocks and certain distinguished $\mathscr{G}$-invariant tensors. Further evidence of an apparent correspondence between classical and quantum maximally symmetric WZW defect junctions is presented in Ref. [RS12].

7. The complete twisted bracket structure, and relative cohomology

In the preceding sections, we have amassed ample evidence in favour of the identification of twisted bracket structures on (twisted) generalised tangent bundles over the target space M and the bi-brane world-volume Q of the σ-model for a world-sheet with circular non-intersecting defect lines as the right differential-algebraic constructs that carry complete information on (infinitesimal) rigid symmetries of the physical theory of interest. Below, we shall complete our description of the generalised geometry of the target space of the multi-phase σ-model for world-sheets with generic defect quivers by adjoining an appropriate structure on the inter-bi-brane world-volume and thus defining an extension of the previously introduced twisted bracket structure to (the generalised tangent bundle over) the composite target space $M \sqcup Q \sqcup T$.

The said extension, while well-justified from the physical vantage point adopted in this paper, may still seem somewhat *ad hoc* to a more mathematically oriented reader. We shall attempt to amend this situation in the second part of the present section by reinterpreting the complete twisted bracket structure in terms of the relative cohomology of the field space of the multi-phase σ-model.

7.1. **The twisted bracket structure for the full string background.** It proves helpful to begin the search for a *natural* completion of the hitherto construction by reappraising the correspondence between generalised tangent bundles equipped with a twisted bracket and generalised tangent bundles twisted by local data of a geometric object (such as, *e.g.*, a gerbe or a circle bundle), equipped with an untwisted bracket. The existence of Hitchin-type isomorphisms between the two structures strongly suggests to regard the underlying geometry as that of a *sheaf*-theoretic extension of the tangent bundle, or – to enable a uniform treatment – of the tangent sheaf $\mathcal{T}\mathscr{M}$ of a given manifold $\mathscr{M}$, *cf.*, *e.g.*, Ref. [Ram04]. The choice of the sheaves to work with is immediately indicated by the cohomological description of the geometric objects entering the definitions of twisted generalised tangent bundles encountered earlier. Thus, we are led to consider the following differential complex:

$$\mathcal{T}^*_\bullet\mathscr{M} \;:\; 0 \xrightarrow{\mathsf{d}^{(-1)}} \mathcal{T}^*_0\mathscr{M} \xrightarrow{\mathsf{d}^{(0)}} \mathcal{T}^*_1\mathscr{M} \xrightarrow{\mathsf{d}^{(1)}} \mathcal{T}^*_2\mathscr{M} \xrightarrow{\mathsf{d}^{(2)}} \cdots \tag{7.1}$$

of differential sheaves:

- $\mathcal{T}^*_0\mathscr{M} := \underline{\mathbb{R}}$, the sheaf of locally constant real-valued functions on $\mathscr{M}$;
- $\mathcal{T}^*_{q+1}\mathscr{M} := \underline{\Omega}^q(\mathscr{M})$, $q \in \mathbb{N}$, the sheaf of locally smooth q-forms on $\mathscr{M}$,

with the coboundary operators given by the zero map[10] $\mathsf{d}^{(-1)}$, the canonical embedding $\mathsf{d}^{(0)} : \underline{\mathbb{R}} \hookrightarrow \underline{\Omega}^0(\mathscr{M})$ and the de Rham differentials $\mathsf{d}^{(q+1)} := \mathsf{d}$. The above complex contains a distinguished sub-complex:

$$\mathsf{T}^*_\bullet\mathscr{M} \;:\; 0 \xrightarrow{\mathsf{d}^{(-1)}} \mathsf{T}^*_0\mathscr{M} \xrightarrow{\mathsf{d}^{(0)}} \mathsf{T}^*_1\mathscr{M} \xrightarrow{\mathsf{d}^{(1)}} \mathsf{T}^*_2\mathscr{M} \xrightarrow{\mathsf{d}^{(2)}} \cdots \tag{7.2}$$

[10] We introduce this map for the sake of consistency of the notation to be used in the remainder of the paper.

composed of

- $\mathsf{T}^*_0\mathscr{M} := \mathbb{R}^{\pi_0(\mathscr{M})}$, the bundle of real-valued functions on $\mathscr{M}$ constant on its connected components (the latter forming the set $\pi_0(\mathscr{M})$);
- $\mathsf{T}^*_{q+1}\mathscr{M} := \Omega^q(\mathscr{M})$, $q \in \mathbb{N}$, the bundle of smooth q-forms on $\mathscr{M}$.

Using these, and the tangent sheaf $\mathcal{T}\mathscr{M}$ of $\mathscr{M}$, we next introduce

Definition 7.1. In the above notation, the **generalised tangent sheaf of type** $(1,q)$ is the direct sum

$$\mathcal{E}^{(1,q)}\mathscr{M} := \mathcal{T}\mathscr{M} \oplus \mathcal{T}^*_q\mathscr{M}\,.$$

It comes with the obvious **anchor (map)**

$$\alpha_{\mathcal{T}\mathscr{M}} \;:\; \mathcal{E}^{(1,q)}\mathscr{M} \to \mathcal{T}\mathscr{M}$$

and the **canonical contraction**

$$(\,\cdot\,,\,\cdot\,)_\lrcorner \quad:\quad \Gamma\big(\mathcal{E}^{(1,q+1)}\mathscr{M}\big) \times \Gamma\big(\mathcal{E}^{(1,q)}\mathscr{M}\big) \to \Gamma\big(\mathcal{T}^*_q\mathscr{M}\big) \;:\; (\mathscr{V} \oplus \upsilon_i, \mathscr{W} \oplus \varpi_i) \mapsto \tfrac{1}{2}\,(\mathscr{V} \lrcorner\, \varpi_i + \mathscr{W} \lrcorner\, \upsilon_i)\,,$$

$$(\,\cdot\,,\,\cdot\,)_\lrcorner \quad:\quad \Gamma\big(\mathcal{E}^{(1,m)}\mathscr{M}\big) \times \Gamma\big(\mathcal{E}^{(1,m)}\mathscr{M}\big) \to \{0\} \;:\; (\mathscr{V} \oplus \upsilon_i, \mathscr{W} \oplus \varpi_i) \mapsto 0\,, \qquad m \in \{0,1\}\,.$$

The sheaf $\mathcal{E}^{(1,q)}\mathscr{M}$ can be endowed with the **Vinogradov bracket** $[\,\cdot\,,\,\cdot\,]^{(q)}_{\rm V}$ defined for $q > 1$ and $q = 1$ as in Eqs. (2.4) and (2.5), respectively, and extended to $\mathcal{E}^{(1,0)}\mathscr{M}$ by embedding the latter in $\mathscr{E}^{(1,1)}\mathscr{M}$, *i.e.* as per

$$[\,\mathscr{V} \oplus c_i\,,\, \mathscr{W} \oplus d_i\,]^{(0)}_{\rm V} = [\mathscr{V}, \mathscr{W}] \oplus 0\,.$$

The bracket for $q \geq 1$ can be twisted by an arbitrary $(q+2)$-form $\mathrm{H}_{(q+2)} \in \Omega^{q+2}(\mathscr{M})$ as in Eq. (3.1).

Upon restriction of the components of $\mathcal{E}^{(1,q)}\mathscr{M}$ to the respective smooth subsheaves, we obtain the **restricted generalised tangent sheaf of type** $(1,q)$

$$\widehat{\mathsf{E}}^{(1,q)}\mathscr{M} := \mathsf{T}\mathscr{M} \oplus \mathsf{T}^*_q\mathscr{M}$$

with the structure inherited from that on $\mathcal{E}^{(1,q)}\mathscr{M}$. On the latter, we may also induce the $\mathrm{H}_{(q+2)}$**-twisted Vinogradov structure**

$$\widehat{\mathfrak{V}}^{(q),\mathrm{H}_{(q+2)}}\mathscr{M} = \Big(\widehat{\mathsf{E}}^{(1,q)}\mathscr{M}, [\,\cdot\,,\,\cdot\,]^{\mathrm{H}_{(q+2)}}_{\rm V}, (\,\cdot\,,\,\cdot\,)_\lrcorner, \alpha_{\mathsf{T}\mathscr{M}}\Big)\,.$$

Clearly, whenever $\mathcal{E}^{(1,q)}$ gives rise to a twisted generalised tangent bundle (in the sense of Definition 2.9), we may require that $\alpha_{\mathsf{T}\mathscr{M}}$ and $(\,\cdot\,,\,\cdot\,)_\lrcorner$ be globally defined, and that $[\,\cdot\,,\,\cdot\,]^{(q)}_{\rm V}$ map pairs of sections into sections, whereby we retrieve the familiar statements of Propositions 2.13 and 2.16. We shall not pursue this issue further. Instead, we consider

Definition 7.2. Adopt the notation of Definition 7.1. Let $\{M, Q, T_n \,|\, n \in \mathbb{N}_{\geq 3}\}$ be a family of smooth manifolds, equipped with a collection of smooth maps

$$\iota_\alpha \;:\; Q \to M\,, \qquad \alpha \in \{1,2\} \qquad\qquad \pi^{k,k+1}_n \;:\; T_n \to Q\,, \qquad k \in \overline{1,n}\,,$$

satisfying the identity

$$\Delta_{T_n} \circ \Delta_Q = 0 \tag{7.3}$$

for Δ_Q and Δ_{T_n} as in Eq. (6.1) (for some fixed collection of signs $\varepsilon^{k,k+1}_n$, $k \in \overline{1,n}$), and with a collection of smooth differential forms $\mathrm{H}_{(3)} \in \Omega^3(M)$, $\mathrm{H}_{(2)} \in \Omega^2(Q)$ and $\mathrm{H}^n_{(1)} \in \Omega^1(T_n)$. Write

$$\mathscr{F} := M \sqcup Q \sqcup \bigsqcup_{n \geq 3} T_n\,.$$

Assume that the forms satisfy the **curvature descent relations**

$$\Delta_Q \mathrm{H}_{(3)} = -\mathsf{d}\mathrm{H}_{(2)}\,, \qquad\qquad \Delta_{T_n}\mathrm{H}_{(2)} = -\mathsf{d}\mathrm{H}^n_{(1)}\,.$$

The $(\iota_\alpha, \pi^{k,k+1}_n)$**-paired restricted generalised tangent sheaves** are defined as

$$\widehat{\mathsf{E}}^{(1,2\sqcup 1\sqcup 0)}\mathscr{F} := \widehat{\mathsf{E}}^{(1,2)}M \sqcup \widehat{\mathsf{E}}^{(1,1)}Q \sqcup \bigsqcup_{n \geq 3} \widehat{\mathsf{E}}^{(1,0)}T_n \to \mathscr{F}\,.$$

We restrict to those sections $\mathfrak{V} = ({}^M\mathscr{V}, {}^Q\mathscr{V}, {}^{T_n}\mathscr{V}) \oplus (\upsilon, \xi, c)$ thereof which are $(\iota_\alpha, \pi^{k,k+1}_n)$**-aligned**, *i.e.* those obeying the conditions

$$\iota_{\alpha\,*}{}^Q\mathscr{V} = {}^M\mathscr{V}\big|_{\iota_\alpha(Q)}\,, \qquad\qquad \pi^{k,k+1}_{n\,*}{}^{T_n}\mathscr{V} = {}^Q\mathscr{V}\big|_{\pi^{k,k+1}_n(T_n)}\,,$$

and which are subject to **section descent equations**

$$\mathsf{d}^{(2)}_{\mathrm{H}_{(3)}}\big({}^M\mathscr{V}\oplus\upsilon\big)=0\,,\qquad \mathsf{d}^{(1)}_{\mathrm{H}_{(2)}}\big({}^Q\mathscr{V}\oplus\xi\big)=-\Delta_Q\upsilon\,,\qquad \mathsf{d}^{(0)}_{\mathrm{H}_{(1)}}\big({}^{T_n}\mathscr{V}\oplus c\big)=-\Delta_{T_n}\xi\,,\tag{7.4}$$

written in terms of the twisted differentials

$$\mathsf{d}^{(q)}_{\mathrm{H}_{(q+1)}}\big(\mathscr{V}\oplus v\big)\equiv\mathsf{d}^{(q)}v+\mathscr{V}\lrcorner\,\mathrm{H}_{(q+1)}\,,$$

where, in particular, $\mathsf{d}^{(0)}c\equiv c$. We denote the set of all these sections as $\Gamma_{(\iota_\alpha,\pi_n^{k,k+1}),\mathsf{d}}\big(\widehat{\mathsf{E}}^{(1,2\sqcup1\sqcup0)}\mathscr{F}\big)$.

The $(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})$**-twisted bracket structure** on $\widehat{\mathsf{E}}^{(1,2\sqcup1\sqcup0)}\mathscr{F}$ is the quadruple

$$\widehat{\mathfrak{M}}^{(2,1,0),(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})}_{(\iota_\alpha,\pi_n^{k,k+1})}:=\Big(\widehat{\mathsf{E}}^{(1,2\sqcup1\sqcup0)}\mathscr{F},[\![\,\cdot\,,\,\cdot\,]\!]^{(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})},(\,\cdot\,,\,\cdot\,)_\lrcorner,\alpha_{\mathsf{T}\mathscr{F}}\Big)\,,$$

with the anchor map and the canonical contraction restricting to the anchor maps and canonical contractions of the component $\mathrm{H}_{(q+2)}$-twisted Vinogradov structures, and with the $(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})$-twisted bracket restricting as

$$\begin{aligned}[\![\,\mathfrak{V},\mathfrak{W}\,]\!]^{(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})}|_{M\sqcup Q}&=[\![\,\mathfrak{V}|_{M\sqcup Q}\,,\,\mathfrak{W}|_{M\sqcup Q}\,]\!]^{(\mathrm{H}_{(3)},\mathrm{H}_{(2)};\Delta_Q)}\,,\\ [\![\,\mathfrak{V},\mathfrak{W}\,]\!]^{(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})}|_{T_n}&=[\,\mathfrak{V}|_{T_n}\,,\,\mathfrak{W}|_{T_n}\,]^{(0)}_{\mathrm{V}}\,,\end{aligned}\tag{7.5}$$

where $[\![\,\cdot\,,\,\cdot\,]\!]^{(\mathrm{H}_{(3)},\mathrm{H}_{(2)};\Delta_Q)}$ is the twisted bracket structure from Definition 5.1.

We find

Proposition 7.3. *In the notation of Definition 7.2, the* $(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})$*-twisted bracket closes on the set* $\Gamma_{(\iota_\alpha,\pi_n^{k,k+1}),\mathsf{d}}\big(\widehat{\mathsf{E}}^{(1,2\sqcup1\sqcup0)}\mathscr{F}\big)$ *of* $(\iota_\alpha,\pi_n^{k,k+1})$*-aligned sections of the restricted generalised tangent sheaves* $\widehat{\mathsf{E}}^{(1,2\sqcup1\sqcup0)}\mathscr{F}$ *subject to the section descent equations* (7.4)*.*

Proof. The only thing that has to be demonstrated is the identity

$$\mathsf{d}^{(0)}_{\mathrm{H}_{(1)}}[\![\,\mathfrak{V},\mathfrak{W}\,]\!]^{(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})}|_{T_n}=-\Delta_{T_n}\mathrm{pr}_{\mathsf{T}^*_1Q}\big([\![\,\mathfrak{V},\mathfrak{W}\,]\!]^{(\mathrm{H}_{(3)},\mathrm{H}_{(2)},\mathrm{H}_{(1)};\Delta_Q,\Delta_{T_n})}|_{M\sqcup Q}\big)\,,$$

which, for $\mathfrak{V}=({}^M\mathscr{V},{}^Q\mathscr{V},{}^{T_n}\mathscr{V})\oplus(\upsilon,\xi,c)$ and $\mathfrak{W}=({}^M\mathscr{W},{}^Q\mathscr{W},{}^{T_n}\mathscr{W})\oplus(\varpi,\zeta,d)$, follows from

$$\begin{aligned}&-\Delta_{T_n}\big({}^Q\mathscr{V}\lrcorner\,\mathsf{d}\zeta-{}^Q\mathscr{W}\lrcorner\,\mathsf{d}\xi+{}^Q\mathscr{V}\lrcorner\,{}^Q\mathscr{W}\lrcorner\,\mathrm{H}_{(2)}+\tfrac{1}{2}\big({}^Q\mathscr{V}\lrcorner\,\Delta_Q\varpi-{}^Q\mathscr{W}\lrcorner\,\Delta_Q\upsilon\big)\big)\\ =\;&{}^{T_n}\mathscr{V}\lrcorner\,\mathsf{d}\big({}^{T_n}\mathscr{W}\lrcorner\,\mathrm{H}_{(1)}\big)-{}^{T_n}\mathscr{W}\lrcorner\,\mathsf{d}\big({}^{T_n}\mathscr{V}\lrcorner\,\mathrm{H}_{(1)}\big)+{}^{T_n}\mathscr{V}\lrcorner\,{}^{T_n}\mathscr{W}\lrcorner\,\mathsf{d}\mathrm{H}_{(1)}\\ =\;&\big[{}^{T_n}\mathscr{V},{}^{T_n}\mathscr{W}\big]\lrcorner\,\mathrm{H}_{(1)}\equiv\mathsf{d}^{(0)}_{\mathrm{H}_{(1)}}\big(\big[{}^{T_n}\mathscr{V},{}^{T_n}\mathscr{W}\big]\oplus0\big)\,.\end{aligned}$$

□

The physical significance of the last result is brought to the fore by the following simple consequence of Propositions 6.1 and 7.3.

Corollary 7.4. *In the notation of Proposition 6.1, infinitesimal rigid symmetries of the two-dimensional non-linear* σ*-model for network-field configurations* $(X\,|\,\Gamma)$ *in string background* $\mathfrak{B}$ *on world-sheet* (Σ,γ) *with a defect quiver* Γ*, as described in Definition I.2.7, correspond to* $(\iota_\alpha,\pi_n^{k,k+1})$*-aligned sections of the restricted generalised tangent sheaves* $\widehat{\mathsf{E}}^{(1,2\sqcup1\sqcup0)}\mathscr{F}$ *from* $\ker\mathrm{pr}_{\mathsf{T}^*_0T_n}$*, subject to the section descent equations with*

$$\mathrm{H}_{(3)}\equiv\mathrm{H}\,,\qquad\mathrm{H}_{(2)}\equiv\omega\,,\qquad\mathrm{H}_{(1)}\equiv0\,,$$

and such that their image under $\alpha_{\mathsf{T}M}$ *is Killing for* g*. Consequently, there exists a restricted* $(\mathrm{H},\omega,0;\Delta_Q,\Delta_{T_n})$*-twisted bracket structure* $\widehat{\mathfrak{M}}^{(2,1,0),(\mathrm{H},\omega,0;\Delta_Q,\Delta_{T_n})}_{(\iota_\alpha,\pi_n^{k,k+1})}$ *on the set of these sections.*

We also readily establish, upon putting together Proposition 5.10 and Corollary 7.4,

Corollary 7.5. *Adopt the notation of Definitions 7.1 and 7.2. The subspace within* $\Gamma(\mathsf{T}\mathscr{F})$ *given by* $\alpha_{\mathsf{T}\mathscr{F}}\Big(\Gamma_{(\iota_\alpha,\pi_n^{k,k+1}),\mathsf{d}}\big(\widehat{\mathsf{E}}^{(1,2\sqcup1\sqcup0)}\mathscr{F}\big)\Big)$ *is a Lie subalgebra, to be denoted as* $\mathfrak{g}_\sigma$*, within the Lie algebra of vector fields on* $\mathscr{F}=M\sqcup Q\sqcup T$ *with a Killing restriction to* M*. Fix a basis* $\{\mathscr{K}_A\}_{A\in\overline{1,\dim\mathfrak{g}_\sigma}}$*, with restrictions* $\mathscr{K}_A|_M={}^M\mathscr{K}_A$*,* $\mathscr{K}_A|_Q={}^Q\mathscr{K}_A$ *and* $\mathscr{K}_A|_{T_n}={}^{T_n}\mathscr{K}_A$ *such that the defining commutation relations*

$$[\mathscr{K}_A,\mathscr{K}_B]=f_{ABC}\,\mathscr{K}_C$$

hold true for some structure constants f_{ABC}. *The corresponding* σ*-symmetric* $(\iota_\alpha, \pi_n^{k,k+1})$*-aligned sections* $\mathfrak{K}_A$ *of the restricted generalised tangent sheaves* $\widehat{\mathsf{E}}^{(1,2\sqcup 1\sqcup 0)}\mathscr{F}$ *with restrictions*

$$\mathfrak{K}_A|_M = {}^M\mathscr{K}_A \oplus \kappa_A =: {}^M\mathfrak{K}_A\,, \qquad \mathfrak{K}_A|_Q = {}^Q\mathscr{K}_A \oplus k_A =: {}^Q\mathfrak{K}_A\,, \qquad \mathfrak{K}_A|_{T_n} = {}^{T_n}\mathscr{K}_A \oplus 0 =: {}^{T_n}\mathfrak{K}_A\,,$$

$$\begin{cases} \mathscr{L}_{{}^M\mathscr{K}_A}\mathrm{g} = 0 \\ \mathsf{d}_{\mathrm{H}}{}^M\mathfrak{K}_A = 0 \end{cases}, \qquad \begin{cases} \iota_{\alpha\,*}{}^Q\mathscr{K}_A = {}^M\mathscr{K}_A|_{\iota_\alpha(Q)} \\ \mathsf{d}_\omega{}^Q\mathfrak{K}_A + \Delta_Q\kappa_A = 0 \end{cases}, \qquad \begin{cases} \pi_{n\,*}^{k,k+1}{}^{T_n}\mathscr{K}_A = {}^Q\mathscr{K}_A|_{\pi_n^{k,k+1}(T_n)} \\ \Delta_{T_n}k_A = 0 \end{cases}$$

and the canonical contraction (with a trivial restriction to $Q \sqcup T$*)*

$$\mathrm{c}_{(AB)} = (\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,)_\lrcorner|_M$$

satisfy the relations

$$[\![\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]\!]^{(\mathrm{H},\omega,0;\Delta_Q,\Delta_{T_n})} = f_{ABC}\,\mathfrak{K}_C + 0 \oplus \alpha_{AB} \tag{7.6}$$

with

$$\alpha_{AB}|_{\mathscr{M}} = \begin{cases} \mathscr{L}_{{}^M\mathscr{K}_A}\kappa_B - f_{ABC}\,\kappa_C - \mathsf{d}\mathrm{c}_{(AB)} & \textit{on} \quad \mathscr{M} = M \\ \mathscr{L}_{{}^Q\mathscr{K}_A}k_B - f_{ABC}\,k_C + \Delta_Q\mathrm{c}_{(AB)} & \textit{on} \quad \mathscr{M} = Q \\ 0 & \textit{on} \quad \mathscr{M} = T_n \end{cases}.$$

7.2. **The relative-cohomological interpretation.** Our derivation of the bracket structure on sections of the restricted generalised tangent sheaves, while essentially devoid of ambiguities, leaves us with a rather non-obvious definition of the $(\mathrm{H},\omega;\Delta_Q)$-twisted bracket, and hence also with an open question as to the underlying algebraic structure. Below, we reinterpret the definition in terms of the relative differential geometry of the target space encoded in the sequence of smooth (inter-)bi-brane maps

$$T \supset T_n \overset{\pi_n^{k,k+1}}{\underset{\cdots}{\rightrightarrows}} Q \xrightarrow{\ \iota_\alpha\ } M \tag{7.7}$$

subject to constraints (I.2.1). The latter immediately suggests extending the standard de Rham complex of $\mathscr{F}$ (resp. its dual) in the direction of structural (*e.g.*, categorial) descent indicated by the arrows in the above diagram. This line of reasoning has found its application in the cohomological discussion of gauge anomalies and inequivalent gaugings presented in Ref. [GSW12, Sec. 11]. Here, we take it up anew with view to elucidating the bracket structure.

The naturalness of the appearance of relative (co)homology in a rigorous description of target-space structures associated with world-sheet defects of the two-dimensional σ-model was pointed up and made clear already in Ref. [KŠ97] and, subsequently, in Refs. [Gaw99, FOS01], where tensorial data of a boundary bi-brane were neatly organised and classified in terms of relative cohomology of the pair (M,D) consisting of the target space M and its distinguished submanifold $\iota_D : D \hookrightarrow M$, identified with the world-volume of a D-brane, that supports a (global) primitive $\omega \in \Omega^2(D)$ of the restricted Kalb–Ramond 3-form $\mathrm{H} \in \Omega^3(M)$ (the gerbe curvature) and thus gives rise to a ι_D^*-relative de Rham 3-cocycle $\mathrm{H} \oplus \omega$,

$$\mathsf{d}^{(3)}_{\iota_D^*}(\mathrm{H} \oplus \omega) := \mathsf{d}\mathrm{H} \oplus (-\mathsf{d}\omega + \iota_D^*\mathrm{H}) = 0\,.$$

The approach pioneered by Klimčík and Ševera was later adapted to the study of a distinguished class of non-boundary bi-branes, with world-volumes $Q \subset M_1 \times M_2$ embedded in the cartesian product of the target spaces M_α, $\alpha \in \{1,2\}$ assigned to the world-sheet patches on either side of the relevant defect line, in Ref. [FSW08] where a gerbe-theoretic description of this class of defects was proposed. Below, we rework the original argument of Fuchs *et al.* in a manner that allows for its generalisation to arbitrary string backgrounds, as introduced in Ref. [RS09].

7.2.1. ***The cohomology for the target space in the presence of defects***. In what follows, we give a construction of a target-space (co)homology underlying the definition of the two-dimensional σ-model in the presence of defects admitting self-intersections. To these ends, we extend to the more general setting of interest (and in the spirit of Ref. [BT82, Sec. 7]) the construction advanced in Ref. [FSW08, App. A].

A natural point of departure in a systematic discussion of the cohomology of the hierarchy of target-space geometries (7.7) and its realisation in terms of differential forms is the introduction of the relevant (singular) homology. Thus, we begin with

Definition 7.6. Let $(M,Q,T_n),\ n\in\mathbb{N}_{\geq 3}$ be a triple of smooth manifolds, equipped with a collection of smooth maps $\iota_\alpha\ :\ Q\to M,\ \alpha\in\{1,2\}$ and $\pi_n^{k,k+1}\ :\ T_n\to Q,\ k\in\mathbb{Z}/n\mathbb{Z}$ subject to the constraints

$$\iota_2^{\varepsilon_n^{k-1,k}}\circ\pi_n^{k-1,k}=\iota_1^{\varepsilon_n^{k,k+1}}\circ\pi_n^{k,k+1}\,,$$

written, for some fixed choice of signs[11] $\varepsilon_n^{k,k+1}$, in the conventions of Definition I.2.1. Moreover, let $C_k(M),C_k(Q)$ and $C_k(T_n)$ be the respective chain groups of the singular chain complexes $C_\bullet(M),C_\bullet(Q)$ and $C_\bullet(T_n)$, equipped with the respective boundary operators $\partial^M_{(k)},\partial^Q_{(k)}$ and $\partial^{T_n}_{(k)}$. Write

$$\Delta^Q:=\iota_{2\,\sharp}-\iota_{1\,\sharp}\,,\qquad\qquad \Delta^{T_n}:=\sum_{k=1}^n\varepsilon_n^{k,k+1}\,\pi^{k,k+1}_{n\,\sharp}$$

for the two combinations of pushforward maps $\iota_{\alpha\,\sharp}$ and the $\pi^{k,k+1}_{n\,\sharp}$ on singular chains induced by the ι_α and the $\pi_n^{k,k+1}$, respectively. The k**-th** (Δ^Q,Δ^{T_n})**-relative chain group** is defined as

$$C_k\left(M,Q,T_n\,|\,\Delta^Q,\Delta^{T_n}\right):=C_k(M)\oplus C_{k-1}(Q)\oplus C_{k-2}(T_n) \tag{7.8}$$

and the associated (Δ^Q,Δ^{T_n})**-relative boundary operators** are given by

$$\begin{aligned}\partial^{(\Delta^Q,\Delta^{T_n})}_{(k)}\quad &:\quad C_k\left(M,Q,T_n\,|\,\Delta^Q,\Delta^{T_n}\right)\to C_{k-1}\left(M,Q,T_n\,|\,\Delta^Q,\Delta^{T_n}\right)\\ &:\quad c^M_k\oplus c^Q_{k-1}\oplus c^{T_n}_{k-2}\mapsto\left(\partial^M_{(k)}c^M_k-\Delta^Q c^Q_{k-1}\right)\oplus\left(-\partial^Q_{(k-1)}c^Q_{k-1}-\Delta^{T_n}c^{T_n}_{k-2}\right)\oplus\partial^{T_n}_{(k-2)}c^{T_n}_{k-2}\,.\end{aligned}$$

They satisfy the fundamental relation

$$\partial^{(\Delta^Q,\Delta^{T_n})}_{(k)}\circ\partial^{(\Delta^Q,\Delta^{T_n})}_{(k+1)}=0\,,$$

and so they give rise to the (Δ^Q,Δ^{T_n})**-relative (singular) chain complex**

$$C_\bullet\left(M,Q,T_n\,|\,\Delta^Q,\Delta^{T_n}\right):=\bigoplus_{k\geq 0}C_k\left(M,Q,T_n\,|\,\Delta^Q,\Delta^{T_n}\right)\,,$$

i.e. the total complex of the semi-bounded bicomplex $C_\bullet(\mathscr{M}_\bullet)$ with $(\mathscr{M}_1,\mathscr{M}_2,\mathscr{M}_3):=(T_n,Q,M)$, defined as

$$\begin{tikzcd}
\cdots \arrow[r, "\partial^{T_n}_{(4)}"] & C_3(T_n) \arrow[r, "\partial^{T_n}_{(3)}"] \arrow[d, "\Delta^{T_n}"] & C_2(T_n) \arrow[r, "\partial^{T_n}_{(2)}"] \arrow[d, "\Delta^{T_n}"] & C_1(T_n) \arrow[r, "\partial^{T_n}_{(1)}"] \arrow[d, "\Delta^{T_n}"] & C_0(T_n) \arrow[r, "\partial^{T_n}_{(0)}"] \arrow[d, "\Delta^{T_n}"] & 0 \\
\cdots \arrow[r, "\partial^{Q}_{(4)}"] & C_3(Q) \arrow[r, "\partial^{Q}_{(3)}"] \arrow[d, "\Delta^{Q}"] & C_2(Q) \arrow[r, "\partial^{Q}_{(2)}"] \arrow[d, "\Delta^{Q}"] & C_1(Q) \arrow[r, "\partial^{Q}_{(1)}"] \arrow[d, "\Delta^{Q}"] & C_0(Q) \arrow[r, "\partial^{Q}_{(0)}"] \arrow[d, "\Delta^{Q}"] & 0 \\
\cdots \arrow[r, "\partial^{M}_{(4)}"] & C_3(M) \arrow[r, "\partial^{M}_{(3)}"] & C_2(M) \arrow[r, "\partial^{M}_{(2)}"] & C_1(M) \arrow[r, "\partial^{M}_{(1)}"] & C_0(M) \arrow[r, "\partial^{M}_{(0)}"] & 0
\end{tikzcd}\,.$$

Its k-th homology group

$$H_k\left(M,Q,T_n\,|\,\Delta^Q,\Delta^{T_n}\right):=\frac{\ker\partial^{(\Delta^Q,\Delta^{T_n})}_{(k)}}{\mathrm{im}\,\partial^{(\Delta^Q,\Delta^{T_n})}_{(k+1)}}$$

will be termed the k**-th** (Δ^Q,Δ^{T_n})**-relative homology group**.

The dual structure is introduced in

[11] Here, we are abusing the original conventions of Ref. [RS09, Sec. 2.5] slightly by denoting the component of T_n corresponding to the fixed choice of signs $\varepsilon_n^{k,k+1}$ with the same symbol.

Definition 7.7. In the notation of Definition 7.6, for $\Delta_Q = \iota_2^* - \iota_1^*$ the dual of Δ^Q, for $\Delta_{T_n} = \sum_{k=1}^n \varepsilon_n^{k,k+1} \pi_n^{k,k+1\,*}$ the dual of Δ^{T_n}, and for R a ring, the **k-th (Δ_Q, Δ_{T_n})-relative cochain group with values in R** is defined as

$$C^k\left(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; R\right) := \mathrm{Hom}_{R-\mathrm{Mod}}\left(C_k\left(M, Q, T_n \,|\, \Delta^Q, \Delta^{T_n}\right), R\right)$$

with $R-$Mod the category of R-modules. The attendant **(Δ_Q, Δ_{T_n})-relative coboundary operators**

$$\delta^{(k)}_{(\Delta_Q,\Delta_{T_n})} \;:\; C^k\left(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; R\right) \to C^{k+1}\left(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; R\right),$$

defined by the duality relations (written for arbitrary $c^k \in C^k\left(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; R\right)$ and $c_{k+1} \in C_{k+1}\left(M, Q, T_n \,|\, \Delta^Q, \Delta^{T_n}\right)$)

$$\delta^{(k)}_{(\Delta_Q,\Delta_{T_n})} c^k\left(c_{k+1}\right) := c^k\left(\partial^{(\Delta^Q,\Delta^{T_n})}_{(k+1)} c_{k+1}\right),$$

determine the **(Δ_Q, Δ_{T_n})-relative cochain complex with values in R**

$$C^\bullet\left(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; R\right) := \bigoplus_{k\geq 0} C^k\left(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; R\right)$$

and its k-th cohomology group

$$H^k\left(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; R\right) := \frac{\ker \delta^{(k)}_{(\Delta_Q,\Delta_{T_n})}}{\mathrm{im}\, \delta^{(k-1)}_{(\Delta_Q,\Delta_{T_n})}},$$

to be termed the **k-th (Δ_Q, Δ_{T_n})-relative (singular) cohomology group with values in R.**

It will prove useful, and – indeed – crucial for the discussion of the situation without defect junctions, to consider also

Definition 7.8. Adopt the notation of Definitions 7.6 and 7.7. The **k-th Δ^Q-relative chain group** is defined as

$$C_k\left(M, Q \,|\, \Delta^Q\right) := C_k(M) \oplus C_{k-1}(Q) \tag{7.9}$$

and the associated **Δ^Q-relative boundary operators** are given by

$$\begin{aligned} \partial^{\Delta^Q}_{(k)} \quad &: \quad C_k\left(M, Q \,|\, \Delta^Q\right) \to C_{k-1}\left(M, Q \,|\, \Delta^Q\right) \\ &: \quad c^M_k \oplus c^Q_{k-1} \mapsto \left(\partial^M_{(k)} c^M_k - \Delta^Q c^Q_{k-1}\right) \oplus \left(-\partial^Q_{(k-1)} c^Q_{k-1}\right). \end{aligned}$$

They satisfy the relation

$$\partial^{\Delta^Q}_{(k)} \circ \partial^{\Delta^Q}_{(k+1)} = 0,$$

and so they give rise to the **Δ^Q-relative (singular) chain complex**

$$C_\bullet\left(M, Q \,|\, \Delta^Q\right) := \bigoplus_{k\geq 0} C_k\left(M, Q \,|\, \Delta^Q\right),$$

i.e. the total complex obtained by truncating the bicomplex $C_\bullet(\mathscr{M}_\bullet)$ of Definition 7.6 as

$$\begin{tikzcd} \cdots \arrow[r, "\partial^Q_{(4)}"] & C_3(Q) \arrow[r, "\partial^Q_{(3)}"] \arrow[d, "\Delta^Q"] & C_2(Q) \arrow[r, "\partial^Q_{(2)}"] \arrow[d, "\Delta^Q"] & C_1(Q) \arrow[r, "\partial^Q_{(1)}"] \arrow[d, "\Delta^Q"] & C_0(Q) \arrow[r, "\partial^Q_{(0)}"] \arrow[d, "\Delta^Q"] & 0 \\ \cdots \arrow[r, "\partial^M_{(4)}"] & C_3(M) \arrow[r, "\partial^M_{(3)}"] & C_2(M) \arrow[r, "\partial^M_{(2)}"] & C_1(M) \arrow[r, "\partial^M_{(1)}"] & C_0(M) \arrow[r, "\partial^M_{(0)}"] & 0 \end{tikzcd}.$$

Its k-th homology group

$$H_k\left(M, Q \,|\, \Delta^Q\right) := \frac{\ker \partial^{\Delta^Q}_{(k)}}{\mathrm{im}\, \partial^{\Delta^Q}_{(k+1)}}$$

will be termed the **k-th Δ^Q-relative homology group**.

Analogously, the **k-th Δ_Q-relative cochain group with values in R** is defined as

$$C^k\left(M, Q \,|\, \Delta_Q; R\right) := \mathrm{Hom}_{R-\mathrm{Mod}}\left(C_k\left(M, Q \,|\, \Delta^Q\right), R\right).$$

The attendant Δ_Q**-relative coboundary operators**

$$\delta^{(k)}_{\Delta_Q} \ :\ C^k\left(M,Q\,|\,\Delta_Q;R\right) \to C^{k+1}\left(M,Q\,|\,\Delta_Q;R\right)\,,$$

defined by the duality relations (written for arbitrary $c^k \in C^k\left(M,Q\,|\,\Delta_Q;R\right)$ and $c_{k+1} \in C_{k+1}\left(M,Q\,|\,\Delta^Q\right)$)

$$\delta^{(k)}_{\Delta_Q} c^k\left(c_{k+1}\right) := c^k\left(\partial^{\Delta^Q}_{(k+1)} c_{k+1}\right)\,,$$

determine the Δ_Q**-relative cochain complex with values in** R

$$C^\bullet\left(M,Q\,|\,\Delta_Q;R\right) := \bigoplus_{k\geq 0} C^k\left(M,Q\,|\,\Delta_Q;R\right)$$

and its k-th cohomology group

$$H^k\left(M,Q\,|\,\Delta_Q;R\right) := \frac{\ker \delta^{(k)}_{\Delta_Q}}{\operatorname{im} \delta^{(k-1)}_{\Delta_Q}}\,,$$

to be termed the k**-th** Δ_Q**-relative (singular) cohomology group with values in** R.

The latter cohomology is characterised in the following important

Proposition 7.9. *In the notation of Definition 7.8, the (singular) cohomology groups $H^k(M;R)$ and $H^k(Q;R)$, and the Δ_Q-relative (singular) cohomology groups $H^k\left(M,Q\,|\,\Delta_Q;R\right)$, all with values in ring R, fit into the long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r, "B^{(k-1)}_{\Delta_Q;R}"] & H^{k-1}(Q;R) \arrow[r] & H^k\left(M,Q\,|\,\Delta_Q;R\right) \arrow[r] & H^k(M;R) \arrow[dlll, "B^{(k)}_{\Delta_Q;R}"'] \\ H^k(Q;R) \arrow[r] & H^{k+1}\left(M,Q\,|\,\Delta_Q;R\right) \arrow[r] & H^{k+1}(M;R) \arrow[r, "B^{(k+1)}_{\Delta_Q;R}"] & \cdots \end{tikzcd} \qquad (7.10)$$

with the connecting homomorphisms

$$B^{(k)}_{\Delta_Q;R} \ :\ H^k(M;R) \to H^k(Q;R) \ :\ [c^k_M] \mapsto [-\Delta^{Q\dagger} c^k_M]$$

defined as

$$\Delta^{Q\dagger} c^k_M(c^Q_k) := c^k_M(\Delta^Q c^Q_k)$$

for $c^Q_k \in C_k(Q)$ arbitrary.

Proof. The cohomology sequence is induced by the short exact sequence of cochain groups

$$0 \to C^{k-1}(Q;R) \xrightarrow{\mathrm{pr}^\dagger_{2\,(k)}} C^k\left(M,Q\,|\,\Delta_Q;R\right) \xrightarrow{\iota^\dagger_{(k)}} C^k(M;R) \to 0\,,$$

whose existence and properties stem from the fact that the (split) short exact sequence of chain groups

$$0 \to C_k(M) \xrightarrow{\iota_{(k)}} C_k\left(M,Q\,|\,\Delta^Q\right) \xrightarrow{\mathrm{pr}_{2\,(k)}} C_{k-1}(Q) \to 0\,,$$

written in terms of the inclusion map $\iota_{(k)}$ and the canonical projection map $\mathrm{pr}_{2\,(k)} \equiv \mathrm{pr}_2$, splits by assumption, *cf.* Eq. (7.9). The former sequence is obtained from the latter one through application of the exact functor $\mathrm{Hom}_{R-\mathrm{Mod}}(\cdot;R)$, and uses the dual maps

$$\begin{aligned} \mathrm{pr}^\dagger_{2\,(k)} c^{k-1}_Q(c^M_k \oplus c^Q_{k-1}) &:= c^{k-1}_Q\left(\mathrm{pr}_{2\,(k)}(c^M_k \oplus c^Q_{k-1})\right) = c^{k-1}_Q(c^Q_{k-1})\,, \\ \iota^\dagger_{(k)} c^k_{M,Q|\Delta_Q}(c^M_k) &:= c^k_{M,Q|\Delta_Q}\left(\iota_{(k)}(c^M_k)\right) = c^k_{M,Q|\Delta_Q}(c^M_k \oplus 0)\,. \end{aligned}$$

Finally, the connecting (Bokshteyn) homomorphism is induced in the usual manner upon noting that every k-cochain c^k_M can be written as

$$c^k_M = \iota^\dagger_{(k)}(c^k_M \circ \mathrm{pr}_1)\,,$$

and whenever it is co-closed, we find

$$\delta^{(k)}_{\Delta_Q}(c^k_M \circ \mathrm{pr}_1) = \mathrm{pr}^\dagger_{2\,(k+1)}(-\Delta^{Q\dagger} c^k_M)\,.$$

$\square$

Similarly, we establish

Proposition 7.10. *In the notation of Definitions 7.7 and 7.8, the (singular) cohomology groups* $H^k(Q;R)$*, the* Δ_Q*-relative (singular) cohomology groups* $H^k(M,Q\,|\,\Delta_Q;R)$ *and the* (Δ_Q,Δ_{T_n})*-relative (singular) cohomology groups* $H^k(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n};R)$*, all with values in ring* R*, fit into the long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r, "B^{(k-1)}_{(\Delta_Q,\Delta_{T_n});R}"] & H^{k-2}(T_n;R) \arrow[r] & H^k(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n};R) \arrow[r] & H^k(M,Q\,|\,\Delta_Q;R) \arrow[dll, "B^{(k)}_{(\Delta_Q,\Delta_{T_n});R}"'] \\ & H^{k-1}(T_n;R) \arrow[r] & H^{k+1}(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n};R) \arrow[r] & H^{k+1}(M,Q\,|\,\Delta_Q;R) \arrow[r, "B^{(k+1)}_{(\Delta_Q,\Delta_{T_n});R}"] & \cdots \end{tikzcd}, \tag{7.11}$$

with the connecting homomorphisms

$$B^{(k)}_{(\Delta_Q,\Delta_{T_n});R}\ :\ H^k(M,Q\,|\,\Delta_Q;R)\to H^{k-1}(T_n;R)\ :\ [c^k_{M,Q|\Delta_Q}]\mapsto[-\Delta^{T_n\,\dagger}_{(2)}(c^k_M\oplus c^{k-1}_Q)]$$

defined as

$$\Delta^{T_n\,\dagger}_{(2)}c^k_{M,Q|\Delta_Q}(c^{T_n}_{k-1}):=c^k_{M,Q|\Delta_Q}(0\oplus\Delta^{T_n}c^{T_n}_{k-1})$$

for $c^{T_n}_{k-1}\in C_{k-1}(T_n)$ *arbitrary.*

Proof. The cohomology sequence is induced by the short exact sequence of cochain groups

$$0\to C^{k-2}(T_n;R)\xrightarrow{\widetilde{\mathrm{pr}}^\dagger_{3\,(k)}}C^k(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n};R)\xrightarrow{\widetilde{\iota}^\dagger_{(k)}}C^k(M,Q\,|\,\Delta_Q;R)\to 0\,,$$

obtained, through application of the exact functor $\mathrm{Hom}_{R\text{-Mod}}(\cdot;R)$, from the (split) short exact sequence of chain groups

$$0\to C_k\left(M,Q\,|\,\Delta^Q\right)\xrightarrow{\widetilde{\iota}_{(k)}}C_k\left(M,Q,T_n\,|\,\Delta^Q,\Delta^{T_n}\right)\xrightarrow{\widetilde{\mathrm{pr}}_{3\,(k)}}C_{k-2}(T_n)\to 0\,,$$

written in terms of the inclusion map $\widetilde{\iota}_{(k)}$ and the canonical projection map $\widetilde{\mathrm{pr}}_{3\,(k)}\equiv\mathrm{pr}_3$. We have the dual maps

$$\begin{aligned}\widetilde{\mathrm{pr}}^\dagger_{3\,(k)}c^{k-2}_{T_n}(c^M_k\oplus c^Q_{k-1}\oplus c^{T_n}_{k-2}) &:= c^{k-2}_{T_n}(c^{T_n}_{k-2})\,,\\ \widetilde{\iota}^\dagger_{(k)}c^k_{M,Q,T_n|\Delta_Q,\Delta_{T_n}}(c^M_k\oplus c^Q_{k-1}) &:= c^k_{M,Q,T_n|\Delta_Q,\Delta_{T_n}}(c^M_k\oplus c^Q_{k-1}\oplus 0)\,.\end{aligned}$$

The definition of the connecting homomorphism is, once again, completely standard since every $c^k_{M,Q|\Delta_Q}$ can be obtained as

$$c^k_{M,Q|\Delta_Q}=\widetilde{\iota}^\dagger_{(k)}(c^k_{M,Q|\Delta_Q}\circ\mathrm{pr}_{1,2})\,,$$

and whenever it is ($\delta^{(k)}_{\Delta_Q}$-)co-closed, we have

$$\delta^{(k)}_{(\Delta_Q,\Delta_{T_n})}(c^k_{M,Q|\Delta_Q}\circ\mathrm{pr}_{1,2})=\widetilde{\mathrm{pr}}^\dagger_{3\,(k)}\left(-\Delta^{T_n\,\dagger}_{(2)}\right)c^k_{M,Q|\Delta_Q}\,.$$

□

It is convenient to have an explicit differential-geometric realisation of the relative cohomologies defined above. The point of departure towards establishing one will be the de Rham isomorphism

$$[I^{(\bullet)}_{\mathscr{M}}]\ :\ H^\bullet_{\mathrm{dR}}(\mathscr{M})\xrightarrow{\cong}H^\bullet(\mathscr{M};\mathbb{R}) \tag{7.12}$$

between the de Rham cohomology and the $\mathbb{R}$-valued singular cohomology, induced from the cochain maps

$$I^{(k)}_{\mathscr{M}}\ :\ \Omega^k(\mathscr{M})\to C^k(\mathscr{M};\mathbb{R})\,,\qquad I^{(k)}_{\mathscr{M}}(\omega^k_{\mathscr{M}})(c^{\mathscr{M}}_k):=\int_{c^{\mathscr{M}}_k}\omega^k_{\mathscr{M}}\,.$$

The only additional element needed for our construction is a relative variant of the de Rham cohomology, to wit,

Definition 7.11. In the notation of Definitions 7.6 and 7.7, and of Eq. (7.1), the **k-th (Δ_Q, Δ_{T_n})-relative de Rham group** is the vector space

$$\begin{aligned} \Omega^k_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) &:= \Omega^k(M) \oplus \Omega^{k-1}(Q) \oplus \Omega^{k-2}(T_n), \quad k \neq 0 \\ \Omega^0_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) &:= \Omega^0(M) \oplus \Omega^{-1}(Q) \end{aligned}$$

with the additional convention that

$$\Omega^{-1}(\mathscr{M}) := \mathbb{R}^{\pi_0(\mathscr{M})}, \quad \mathscr{M} \in \{Q, T_n\}.$$

The associated **(Δ_Q, Δ_{T_n})-relative coboundary operators**

$$\begin{aligned} \mathsf{d}^{(k)}_{(\Delta_Q, \Delta_{T_n})} &: \Omega^k_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q) \to \Omega^{k+1}_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q, D_{T_n}) \\ &: \omega^k_M \oplus \omega^{k-1}_Q \oplus \omega^{k-2}_{T_n} \mapsto \mathsf{d}^{(k)}\omega^k_M \oplus (-\mathsf{d}^{(k-1)}\omega^{k-1}_Q - \Delta_Q \omega^k_M) \oplus (\mathsf{d}^{(k-2)}\omega^{k-2}_{T_n} - \Delta_{T_n}\omega^{k-1}_Q) \end{aligned}$$

yield the **(Δ_Q, Δ_{T_n})-relative de Rham complex**[12]

$$\Omega^\bullet_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) := \bigoplus_{k \geq 0} \Omega^k_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}),$$

that is the total complex of the semi-bounded bicomplex $\Omega^\bullet(\mathscr{M}_\bullet)$, and its k-th cohomology group

$$H^k_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) := \frac{\ker \mathsf{d}^{(k)}_{(\Delta_Q, \Delta_{T_n})}}{\operatorname{im} \mathsf{d}^{(k-1)}_{(\Delta_Q, \Delta_{T_n})}}$$

to be termed the **k-th (Δ_Q, Δ_{T_n})-relative de Rham cohomology group**.

Its truncated version is given in

Definition 7.12. In the notation of Definitions 7.8 and 7.11, and of Eq. (7.1), the **k-th Δ_Q-relative de Rham group** is the vector space

$$\Omega^k_{\mathrm{dR}}(M, Q \,|\, \Delta_Q) := \Omega^k(M) \oplus \Omega^{k-1}(Q).$$

The associated **Δ_Q-relative coboundary operators**

$$\begin{aligned} \mathsf{d}^{(k)}_{\Delta_Q} &: \Omega^k_{\mathrm{dR}}(M, Q \,|\, \Delta_Q) \to \Omega^{k+1}_{\mathrm{dR}}(M, Q \,|\, \Delta_Q) \\ &: \omega^k_M \oplus \omega^{k-1}_Q \mapsto \mathsf{d}^{(k)}\omega^k_M \oplus (-\mathsf{d}^{(k-1)}\omega^{k-1}_Q - \Delta_Q \omega^k_M) \end{aligned}$$

yield the **Δ_Q-relative de Rham complex**[13]

$$\Omega^\bullet_{\mathrm{dR}}(M, Q \,|\, \Delta_Q) := \bigoplus_{k \geq 0} \Omega^k_{\mathrm{dR}}(M, Q \,|\, \Delta_Q)$$

and its k-th cohomology group

$$H^k_{\mathrm{dR}}(M, Q \,|\, \Delta_Q) := \frac{\ker \mathsf{d}^{(k)}_{\Delta_Q}}{\operatorname{im} \mathsf{d}^{(k-1)}_{\Delta_Q}}$$

to be termed the **k-th Δ_Q-relative de Rham cohomology group**.

We have the following relative counterpart of the de Rham Theorem.

Theorem 7.13. [FSW08, App. A] *Adopt the notation of Definitions 7.6, 7.7, 7.8 and 7.12. The $\mathbb{R}$-linear map*

$$I^{(k)}_{\Delta_Q} : \Omega^k_{\mathrm{dR}}(M, Q \,|\, \Delta_Q) \to C^k(M, Q \,|\, \Delta_Q; \mathbb{R})$$

defined by the formula

$$I^{(k)}_{\Delta_Q}(\omega^k_M \oplus \omega^{k-1}_Q)(c^M_k \oplus c^Q_{k-1}) := \int_{c^M_k} \omega^k_M + \int_{c^Q_{k-1}} \omega^{k-1}_Q,$$

[12] We shall occasionally use the same name for the pair $\left(\Omega^k_{\mathrm{dR}}(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}), \mathsf{d}^{(k)}_{(\Delta_Q, \Delta_{T_n})}\right)$.

[13] We shall occasionally use the same name for the pair $\left(\Omega^k_{\mathrm{dR}}(M, Q \,|\, \Delta_Q), \mathsf{d}^{(k)}_{\Delta_Q}\right)$.

written for an arbitrary Δ^Q-relative k-chain $c_k^M \oplus c_{k-1}^Q \in C_k\left(M,Q\,|\,\Delta^Q\right)$, is a cochain map. The induced homomorphism

$$[I^{(k)}_{\Delta_Q}] \;:\; H^k_{\mathrm{dR}}(M,Q\,|\,\Delta_Q) \to H^k(M,Q\,|\,\Delta_Q;\mathbb{R}) \tag{7.13}$$

is a group isomorphism.

Proof. That $I^{(k)}_{\Delta_Q}$ is a cochain map readily follows from direct computation,

$$\begin{aligned}
\delta^{(k)}_{\Delta_Q}\left(I^{(k)}_{\Delta_Q}(\omega_M^k \oplus \omega_Q^{k-1})\right)(c^M_{k+1} \oplus c^Q_k) &\equiv I^{(k)}_{\Delta_Q}(\omega_M^k \oplus \omega_Q^{k-1})\left(\left(\partial^M_{(k+1)}c^M_{k+1} - \Delta^Q c^Q_k\right) \oplus \left(-\partial^Q_{(k)}c^Q_k\right)\right) \\
&= \int_{\partial^M_{(k+1)}c^M_{k+1} - \Delta^Q c^Q_k} \omega^k_M - \int_{\partial^Q_{(k)}c^Q_k} \omega^{k-1}_Q \\
&= \int_{c^M_{k+1}} \mathsf{d}^{(k)}\omega^k_M + \int_{c^Q_k}\left(-\mathsf{d}^{(k-1)}\omega^{k-1}_Q - \Delta_Q\omega^k_M\right) \\
&\equiv I^{(k)}_{\Delta_Q}\left(\mathsf{d}^{(k)}_{\Delta_Q}(\omega^k_M \oplus \omega^{k-1}_Q)\right)(c^M_{k+1} \oplus c^Q_k)\,.
\end{aligned}$$

We may, next, use the split exact sequence (existing by construction)

$$0 \to \Omega^{k-1}(Q) \xrightarrow{\iota^{(k)}_{\mathrm{dR}}} \Omega^k_{\mathrm{dR}}(M,Q\,|\,\Delta_Q) \xrightarrow{\mathrm{pr}^{(k)}_{1\,\mathrm{dR}}} \Omega^k(M) \to 0\,,$$

expressed in terms of the inclusion map $\iota^{(k)}_{\mathrm{dR}}$ and the canonical projection $\mathrm{pr}^{(k)}_{1\,\mathrm{dR}} \equiv \mathrm{pr}_1$, to induce the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\beta^{(k-1)}_{\Delta_Q}"] & H^{k-1}_{\mathrm{dR}}(Q) \arrow[r] & H^k_{\mathrm{dR}}(M,Q\,|\,\Delta_Q) \arrow[r] & H^k_{\mathrm{dR}}(M) \arrow[dll, "\beta^{(k)}_{\Delta_Q}"'] \\
& H^k_{\mathrm{dR}}(Q) \arrow[r] & H^{k+1}_{\mathrm{dR}}(M,Q\,|\,\Delta_Q) \arrow[r] & H^{k+1}_{\mathrm{dR}}(M) \arrow[r, "\beta^{(k+1)}_{\Delta_Q}"] & \cdots
\end{tikzcd}, \tag{7.14}$$

with the (standard) connecting homomorphisms

$$\beta^{(k)}_{\Delta_Q} \;:\; H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(Q) \;:\; [\omega^k_M] \mapsto [-\Delta_Q\omega^k_M]\,.$$

In conjunction with the long exact sequence of Eq. (7.10), it gives rise to the manifestly commutative diagram with exact rows

$$\begin{tikzcd}
H^{k-1}_{\mathrm{dR}}(M) \arrow[r, "\beta^{(k-1)}_{\Delta_Q}"] \arrow[d, "{[I^{(k-1)}_M]}"] & H^{k-1}_{\mathrm{dR}}(Q) \arrow[r, "{[\iota^{(k)}_{\mathrm{dR}}]}"] \arrow[d, "{[I^{(k-1)}_Q]}"] & H^k_{\mathrm{dR}}(M,Q\,|\,\Delta_Q) \arrow[r, "{[\mathrm{pr}^{(k)}_{1\,\mathrm{dR}}]}"] \arrow[d, "{[I^{(k)}_{\Delta_Q}]}"] & H^k_{\mathrm{dR}}(M) \arrow[r, "\beta^{(k)}_{\Delta_Q}"] \arrow[d, "{[I^{(k)}_M]}"] & H^k_{\mathrm{dR}}(Q) \arrow[d, "{[I^{(k)}_Q]}"] \\
H^{k-1}(M;\mathbb{R}) \arrow[r, "B^{(k-1)}_{\Delta_Q;\mathbb{R}}"'] & H^{k-1}(Q;\mathbb{R}) \arrow[r, "{[\mathrm{pr}^\dagger_{2\,(k)}]}"'] & H^k(M,Q\,|\,\Delta_Q;\mathbb{R}) \arrow[r, "{[\iota^\dagger_{(k)}]}"'] & H^k(M;\mathbb{R}) \arrow[r, "B^{(k)}_{\Delta_Q;\mathbb{R}}"'] & H^k(Q;\mathbb{R})
\end{tikzcd},$$

in which the $[I^{(k)}_{\mathscr{M}}]$ are as in Eq. (7.12), and all maps in rectangular brackets are defined as the cohomology lifts of the respective cochain maps, *e.g.*,

$$[\mathrm{pr}^\dagger_{1\,(k)}][c^{k-1}_Q] \coloneqq [\mathrm{pr}^\dagger_{1\,(k)}c^{k-1}_Q]\,.$$

Since the $[I^{(k)}_{\mathscr{M}}]$ are isomorphisms, the commutativity of the above diagram immediately implies, in virtue of the Five Lemma of Ref. [ML75, Lemma I.3.3], that $[I^{(k)}_{\Delta_Q}]$ is, indeed, an isomorphism. □

The last theorem is instrumental in proving its own extended version:

Theorem 7.14. *Adopt the notation of Definitions 7.6, 7.7 and 7.11. The $\mathbb{R}$-linear map*

$$I^{(k)}_{(\Delta_Q,\Delta_{T_n})} \;:\; \Omega^k(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n}) \to C^k(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n};\mathbb{R})$$

defined by the formula

$$I^{(k)}_{(\Delta_Q,\Delta_{T_n})}(\omega^k_M \oplus \omega^{k-1}_Q \oplus \omega^{k-2}_{T_n})(c^M_k \oplus c^Q_{k-1} \oplus c^{T_n}_{k-2}) \coloneqq \int_{c^M_k}\omega^k_M + \int_{c^Q_{k-1}}\omega^{k-1}_Q + \int_{c^{T_n}_{k-2}}\omega^{k-2}_{T_n}\,,$$

written for an arbitrary (Δ^Q, Δ^{T_n})-relative k-chain $c_k^M \oplus c_{k-1}^Q \oplus c_{k-2}^{T_n} \in C_k\big(M, Q, T_n \,|\, \Delta^Q, \Delta^{T_n}\big)$, is a cochain map. The induced homomorphism

$$[I^{(k)}_{(\Delta_Q,\Delta_{T_n})}] \,:\, H^k_{\rm dR}\,(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) \to H^k\,(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; \mathbb{R}) \tag{7.15}$$

is a group isomorphism.

Proof. That the $I^{(k)}_{(\Delta_Q,\Delta_{T_n})}$ are cochain maps follows from a similar calculation as for the $I^{(k)}_{\Delta_Q}$. It therefore remains to verify that the induced cohomology maps are isomorphisms. Here, we consider the long exact sequence

$$\begin{array}{l} \cdots \xrightarrow{\beta^{(k-1)}_{(\Delta_Q,\Delta_{T_n})}} H^{k-2}_{\rm dR}(T_n) \longrightarrow H^k_{\rm dR}\,(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) \longrightarrow H^k_{\rm dR}(M, Q \,|\, \Delta_Q) \xrightarrow{\beta^{(k)}_{(\Delta_Q,\Delta_{T_n})}} \\ H^{k-1}_{\rm dR}(T_n) \longrightarrow H^{k+1}_{\rm dR}\,(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) \longrightarrow H^{k+1}_{\rm dR}(M, Q \,|\, \Delta_Q) \xrightarrow{\beta^{(k+1)}_{(\Delta_Q,\Delta_{T_n})}} \cdots \end{array} \tag{7.16}$$

induced by the split exact sequence

$$0 \to \Omega^{k-2}(T_n) \xrightarrow{\widetilde{\iota}^{(k)}_{\rm dR}} \Omega^k_{\rm dR}\,(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) \xrightarrow{\widetilde{\rm pr}^{(k)}_{1,2\,{\rm dR}}} \Omega^k_{\rm dR}(M, Q \,|\, \Delta_Q) \to 0$$

in which $\widetilde{\iota}^{(k)}_{\rm dR}$ is the inclusion and $\widetilde{\rm pr}^{(k)}_{1,2\,{\rm dR}} \equiv {\rm pr}_{1,2}$ is the canonical projection. Above, $\beta^{(k)}_{(\Delta_Q,\Delta_{T_n})}$ is the (standard) connecting homomorphisms given by

$$\beta^{(k)}_{(\Delta_Q,\Delta_{T_n})} \,:\, H^k_{\rm dR}(M, Q \,|\, \Delta_Q) \to H^{k-1}_{\rm dR}(T_n) \,:\, [\omega^k_M \oplus \omega^{k-1}_Q] \mapsto [-\Delta_{T_n}\omega^{k-1}_Q]\,.$$

The long exact sequences (7.11) and (7.16) altogether yield the manifestly commutative diagram with exact columns

$$\begin{tikzcd}
H^{k-1}_{\rm dR}(M, Q \,|\, \Delta_Q) \arrow[r, "{[I^{(k-1)}_{\Delta_Q}]}"] \arrow[d, "\beta^{(k-1)}_{(\Delta_Q,\Delta_{T_n})}"'] & H^{k-1}(M, Q \,|\, \Delta_Q; \mathbb{R}) \arrow[d, "B^{(k-1)}_{(\Delta_Q,\Delta_{T_n});\mathbb{R}}"] \\
H^{k-2}_{\rm dR}(T_n) \arrow[r, "{[I^{(k-2)}_{T_n}]}"] \arrow[d, "{[\widetilde{\iota}^{(k)}_{\rm dR}]}"'] & H^{k-2}(T_n; \mathbb{R}) \arrow[d, "{[\widetilde{\rm pr}^{\dagger}_{3\,(k)}]}"] \\
H^k_{\rm dR}\,(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}) \arrow[r, "{[I^{(k)}_{(\Delta_Q,\Delta_{T_n})}]}"] \arrow[d, "{[\widetilde{\rm pr}^{(k)}_{1,2\,{\rm dR}}]}"'] & H^k\,(M, Q, T_n \,|\, \Delta_Q, \Delta_{T_n}; \mathbb{R}) \arrow[d, "{[\widetilde{\iota}^{\dagger}_{(k)}]}"] \\
H^k_{\rm dR}(M, Q \,|\, \Delta_Q) \arrow[r, "{[I^{(k)}_{\Delta_Q}]}"] \arrow[d, "\beta^{(k)}_{(\Delta_Q,\Delta_{T_n})}"'] & H^k(M, Q \,|\, \Delta_Q; \mathbb{R}) \arrow[d, "B^{(k)}_{(\Delta_Q,\Delta_{T_n});\mathbb{R}}"] \\
H^{k-1}_{\rm dR}(T_n) \arrow[r, "{[I^{(k-1)}_{T_n}]}"] & H^{k-1}(T_n; \mathbb{R})
\end{tikzcd},$$

in which both the $[I^{(k)}_{T_n}]$ and the $[I^{(k)}_{\Delta_Q}]$ are isomorphisms, the latter by Theorem 7.13, and all maps in rectangular brackets are defined as the cohomology lifts of the respective cochain maps. Adducing the Five Lemma once more, we conclude that the $[I^{(k)}_{(\Delta_Q,\Delta_{T_n})}]$ are also isomorphisms, as claimed. □

7.2.2. ***The relative Cartan calculus and the twisted bracket.*** The replacement of the standard de Rham cohomology by its relative counterpart in the presence of world-sheet defects and the associated (inter-)bi-brane extension of the string background of Definition I.2.1 suggests that we reconsider the concept of a twisted Courant bracket in the relative-geometric framework. Indeed, the latter concept is based on two differential-geometric structures present on the target space, namely the Lie algebra of vector fields and the de Rham complex of forms that determines, through Cartan's magic formula, the form component of the bracket. Taking as the point of departure the geometry of the target space $\mathscr{F}$ of the background with bi-branes and inter-bi-branes, the respective structures on the target space M, on the bi-brane world-volume Q and on the (component) inter-bi-brane world-volumes T_n become related

by the $(\iota_\alpha,\pi_n^{k,k+1})$-alignment condition (6.3) and by the (Δ_Q,Δ_{T_n})-twist in the de Rham complex. It therefore seems pertinent to enquire as to a natural definition of the Courant bracket, with a twist now determined by the pair (H,ω), in this constrained setting.

We start by giving a relative variant of the Cartan calculus for the coupled target-space geometries (7.7).

Definition 7.15. Adopt the notation of Definitions 7.6, 7.7 and 7.11. Denote the space $\Gamma(\mathsf{T}M\sqcup\mathsf{T}Q\sqcup\mathsf{T}T)$ of vector fields on $M\sqcup Q\sqcup T$, $T=\bigsqcup_{n\geq 3}T_n$ with restrictions $\mathscr{V}|_M={}^M\mathscr{V}, \mathscr{V}|_Q={}^Q\mathscr{V}$ and $\mathscr{V}|_{T_n}={}^{T_n}\mathscr{V}$ satisfying the $(\iota_\alpha,\pi_n^{k,k+1})$-alignment condition (6.3) as $\mathfrak{X}_{(\iota_\alpha,\pi_n^{k,k+1})}(M\sqcup Q\sqcup T)$. To every such vector field $\mathscr{V}\in\mathfrak{X}_{(\iota_\alpha,\pi_n^{k,k+1})}(M\sqcup Q\sqcup T)$ we associate a degree-(-1) derivation of the (Δ_Q,Δ_{T_n})-relative de Rham complex

$$\begin{aligned}\imath_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})}\quad &:\quad \Omega^\bullet_{\mathrm{dR}}(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n})\to\Omega^{\bullet-1}_{\mathrm{dR}}(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n})\\ &:\quad \omega_M^k\oplus\omega_Q^{k-1}\oplus\omega_{T_n}^{k-2}\mapsto\left({}^M\mathscr{V}\lrcorner\,\omega_M^k\right)\oplus\left(-{}^Q\mathscr{V}\lrcorner\,\omega_Q^{k-1}\right)\oplus\left({}^{T_n}\mathscr{V}\lrcorner\,\omega_{T_n}^{k-2}\right),\end{aligned}$$

to be termed the (Δ_Q,Δ_{T_n})**-relative contraction** henceforth.

The (Δ_Q,Δ_{T_n})**-relative Lie derivative on** (Δ_Q,Δ_{T_n})**-relative de Rham complex along** $(\iota_\alpha,\pi_n^{k,k+1})$**-aligned vector field** $\mathscr{V}$ is defined by Cartan's magic formula

$$\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})}|_{\Omega^k_{\mathrm{dR}}(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n})}\coloneqq\mathsf{d}^{(k-1)}_{(\Delta_Q,\Delta_{T_n})}\circ\imath_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})\,(k)}+\imath_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})\,(k+1)}\circ\mathsf{d}^{(k)}_{(\Delta_Q,\Delta_{T_n})}.$$

Remark 7.16. The definition of the (Δ_Q,Δ_{T_n})-relative Lie derivative is not only natural but also yields a simple object when calculated explicitly,

$$\begin{aligned}\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})}\left(\omega_M^k\oplus\omega_Q^{k-1}\oplus\omega_{T_n}^{k-2}\right) &= \mathscr{L}_{{}^M\mathscr{V}}\omega_M^k\oplus\left(\mathscr{L}_{{}^Q\mathscr{V}}\omega^{k-1}-\Delta_Q\left({}^M\mathscr{V}\lrcorner\,\omega_M^k\right)+{}^Q\mathscr{V}\lrcorner\,\Delta_Q\omega_M^k\right)\\ &\quad\oplus\left(\mathscr{L}_{{}^{T_n}\mathscr{V}}\omega_{T_n}^{k-2}+\Delta_{T_n}\left({}^Q\mathscr{V}\lrcorner\,\omega_Q^{k-1}\right)-{}^{T_n}\mathscr{V}\lrcorner\,\Delta_{T_n}\omega_Q^{k-1}\right)\\ &= \mathscr{L}_{{}^M\mathscr{V}}\omega_M^k\oplus\mathscr{L}_{{}^Q\mathscr{V}}\omega_Q^{k-1}\oplus\mathscr{L}_{{}^{T_n}\mathscr{V}}\omega_{T_n}^{k-2}.\end{aligned}$$

It ought to be emphasised that the form of the (Δ_Q,Δ_{T_n})-relative differential is essentially fixed by that of the (Δ_Q,Δ_{T_n})-relative boundary operator, and so we may regard the above observation as a rationale for the definition of the (Δ_Q,Δ_{T_n})-relative contraction and of the (Δ_Q,Δ_{T_n})-relative Lie derivative.

We readily establish

Proposition 7.17. *In the notation of Definitions 7.11 and 7.15, the triple*

$$\left(\mathsf{d}_{(\Delta_Q,\Delta_{T_n})},\imath_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})},\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})}\right),$$

with $\mathsf{d}_{(\Delta_Q,\Delta_{T_n})}\coloneqq(\mathsf{d}^{(k)}_{(\Delta_Q,\Delta_{T_n})})$, *obeys the standard rules of Cartan's calculus:*

$$\mathsf{d}^2_{(\Delta_Q,\Delta_{T_n})}=0\,,\qquad \left\{\imath_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})},\imath_{\mathscr{W}}^{(\Delta_Q,\Delta_{T_n})}\right\}=0\,,\qquad \left[\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})},\mathscr{L}_{\mathscr{W}}^{(\Delta_Q,\Delta_{T_n})}\right]=\mathscr{L}_{[\mathscr{V},\mathscr{W}]}^{(\Delta_Q,\Delta_{T_n})},$$

$$\left\{\mathsf{d}_{(\Delta_Q,\Delta_{T_n})},\imath_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})}\right\}=\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})},$$

$$\left[\mathsf{d}_{(\Delta_Q,\Delta_{T_n})},\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})}\right]=0\,,\qquad \left[\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})},\imath_{\mathscr{W}}^{(\Delta_Q,\Delta_{T_n})}\right]=\imath_{[\mathscr{V},\mathscr{W}]}^{(\Delta_Q,\Delta_{T_n})}.$$

Proof. Obvious, through inspection. □

Thus, we may think of the triple $\left(\mathsf{d}_{(\Delta_Q,\Delta_{T_n})},\imath_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})},\mathscr{L}_{\mathscr{V}}^{(\Delta_Q,\Delta_{T_n})}\right)$ as a natural counterpart of the standard triple $(\mathsf{d},\mathscr{V}\lrcorner,\mathscr{L}_{\mathscr{V}})$ in the setting of coupled target-space geometries. It is now straightforward to consider the corresponding notion of a (twisted) Courant bracket.

Definition 7.18. In the notation of Definitions 7.6, 7.7, 7.11 and 7.15, and for an arbitrary $\mathsf{d}_{(\Delta_Q,\Delta_{T_n})}$-closed (Δ_Q,Δ_{T_n})-relative 3-form $\eta\in\Omega^3_{\mathrm{dR}}(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n})$, we define the η**-twisted** (Δ_Q,Δ_{T_n})**-relative Courant bracket on**

$$\mathfrak{E}_{(\iota_\alpha,\pi_n^{k,k+1})}(M\sqcup Q\sqcup T)\coloneqq\mathfrak{X}_{(\iota_\alpha,\pi_n^{k,k+1})}(M\sqcup Q\sqcup T)\oplus\Omega^1_{\mathrm{dR}}(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n})$$

by the formula

$$[\mathscr{V}\oplus\upsilon,\mathscr{W}\oplus\varpi]^\eta_{\mathrm{C}} := [\mathscr{V},\mathscr{W}]\oplus\Big(\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{V}}\varpi-\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{W}}\upsilon-\tfrac{1}{2}\,\mathsf{d}^{(1)}_{(\Delta_Q,\Delta_{T_n})}\big(\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{V}}\varpi-\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{W}}\upsilon\big) +\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{V}}\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{W}}\eta\Big).$$

The adequacy of the (Δ_Q,Δ_{T_n})-relative Cartan calculus in the discussion of string backgrounds with bi-branes and inter-bi-branes is illustrated amply by the following theorem, which – at the same time – demystifies the previous definition of the $(\mathrm{H},\omega;\Delta_Q)$-twisted bracket structure on $(\iota_\alpha,\pi_n^{k,k+1})$-paired restricted tangent sheaves.

Theorem 7.19. *Adopt the notation of Definitions 7.1, 7.2, 7.11, 7.15 and 7.18, and of Corollary 7.4. The* (Δ_Q,Δ_{T_n})*-relative 3-form* $\mathrm{H}\oplus\omega\oplus 0$ *on* $M\sqcup Q\sqcup T=\mathscr{F}$ *is* $\mathsf{d}_{(\Delta_Q,\Delta_{T_n})}$*-closed, and hence defines a* $(\mathrm{H}\oplus\omega\oplus 0)$*-twisted* (Δ_Q,Δ_{T_n})*-relative Courant bracket on* $\mathfrak{E}_{(\iota_\alpha,\pi_n^{k,k+1})}(\mathscr{F})$*. Under the natural identification between the latter space and* $\Gamma_{(\iota_\alpha,\pi_n^{k,k+1})}\big(\widehat{\mathsf{E}}^{(1,2\sqcup 1\sqcup 0)}\mathscr{F}\big)$ *(expressed in terms of the canonical projections to the direct summands of* $\Omega^1_{\mathrm{dR}}(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n})=\Omega^1(M)\oplus\Omega^0(Q)\oplus\Omega^{-1}(T_n)$*)*

$$\begin{aligned}\Psi\ &:\ \mathfrak{E}_{(\iota_\alpha,\pi_n^{k,k+1})}(\mathscr{F})\xrightarrow{\ \cong\ }\Gamma_{(\iota_\alpha,\pi_n^{k,k+1})}\big(\widehat{\mathsf{E}}^{(1,2\sqcup 1\sqcup 0)}\mathscr{F}\big)\\ &:\ \mathscr{V}\oplus\upsilon\mapsto\big(\mathscr{V}|_M\oplus\mathrm{pr}_1(\upsilon),\mathscr{V}|_Q\oplus\mathrm{pr}_2(\upsilon),\mathscr{V}|_{T_n}\oplus\mathrm{pr}_3(\upsilon)\big)\,,\end{aligned}$$

we have

$$[\![\,\cdot\,,\,\cdot\,]\!]^{(\mathrm{H},\omega,0;\Delta_Q,\Delta_{T_n})}\circ(\Psi,\Psi)=\Psi\circ[\cdot,\cdot]^{\mathrm{H}\oplus\omega\oplus 0}_{\mathrm{C}}\,.$$

Furthermore, the σ-symmetric sections in $\Gamma_{(\iota_\alpha,\pi_n^{k,k+1})}\big(\widehat{\mathsf{E}}^{(1,2\sqcup 1\sqcup 0)}\mathscr{F}\big)$ *are identified with those elements of* $\mathfrak{E}_{(\iota_\alpha,\pi_n^{k,k+1})}(\mathscr{F})$ *that satisfy the relation*

$$\mathsf{d}_{(\Delta_Q,\Delta_{T_n}),\mathrm{H}\oplus\omega\oplus 0}(\mathscr{V}\oplus\upsilon):=\mathsf{d}^{(1)}_{(\Delta_Q,\Delta_{T_n})}\upsilon+\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{V}}(\mathrm{H}\oplus\omega\oplus 0)=0\,. \tag{7.17}$$

Proof. Obvious, through inspection. □

Remark 7.20. It deserves to be emphasised that the above formalism restricts in just the desired manner to the paired geometries (M,Q), *i.e.* in the absence of inter-bi-branes.

7.2.3. ***A proof of Proposition 5.11***. The preceding considerations provide us with cohomological tools necessary for verifying the thesis of Proposition 5.11 that will, in turn, prove central to the discussion of the small gauge anomaly in the framework of generalised geometry in Section 8.

Taking into account Eq. (5.30) and identity (5.17), we can explicitly write out Eq. (5.28) in the basis $\mathfrak{K}_A$ as

$$\begin{aligned}\{h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_A},h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_B}\}_{\Omega_{\sigma,\mathcal{B}|(\pi,\varepsilon)}} &= f_{ABC}\,h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_C}+\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*}\big({}^M\Delta_{AB}-\mathsf{d}\mathrm{c}_{(AB)}\big)+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*}\big({}^Q\Delta_{AB}+\Delta_Q\mathrm{c}_{(AB)}\big)\\ &= f_{ABC}\,h^{\mathcal{B}|\varepsilon}_{\mathfrak{K}_C}+\widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*\,M}\Delta_{AB}+\varepsilon\,\widetilde{\underline{\mathsf{L}}}^{Q|(\pi,\varepsilon)\,*\,Q}\Delta_{AB}\,,\end{aligned}$$

where in the last line we used the notation of Eq. (3.5), with additional abbreviations

$${}^M\Delta_{AB}:=\mathscr{L}_{{}^M\mathscr{K}_A}\kappa_B-f_{ABC}\,\kappa_C\,,\qquad\qquad {}^Q\Delta_{AB}:=\mathscr{L}_{{}^Q\mathscr{K}_A}k_B-f_{ABC}\,k_C\,.$$

The realisation of $\mathfrak{g}_\sigma$ is hamiltonian iff the identity

$$\big(\mathsf{L}^{Q|(\pi,\varepsilon)\,*\,M}\Delta_{AB}+\varepsilon\,\underline{\mathsf{L}}^{Q|(\pi,\varepsilon)\,*\,Q}\Delta_{AB}\big)[(X,q)]=0 \tag{7.18}$$

obtains for every 1-twisted loop (X,q). Write

$$c_1^M(X,q):=X(\mathbb{S}^1_\pi)\,,\qquad\qquad c_0^Q(X,q):=X(\pi)=q\,.$$

The chain $(c_1^M\oplus\varepsilon\,c_0^Q)(X,q)=:c_1^{\Delta^Q}(X,q)$ defines a Δ^Q-relative 1-cycle, and – clearly – any such 1-cycle can be obtained within $\mathsf{L}_{Q|(\pi,\varepsilon)}M$. We may now rewrite condition (7.18) as

$$\langle[I^{(1)}_{\Delta_Q}][{}^M\Delta_{AB}\oplus{}^Q\Delta_{AB}],[c_1^{\Delta^Q}(X,q)]\rangle=0$$

in terms of the induced isomorphism of Eq. (7.13) and of the standard pairing

$$\langle\cdot,\cdot\rangle\ :\ H^1(M,Q\,|\,\Delta_Q;\mathbb{R})\times H_1\big(M,Q\,|\,\Delta^Q\big)\to\mathbb{R}\ :\ \big([c^1_{\Delta_Q}],[c_1^{\Delta^Q}]\big)\mapsto c^1_{\Delta_Q}\big(c_1^{\Delta^Q}\big)\,.$$

In view of the arbitrariness of $c_1^{\Delta^Q}(X,q)$, we conclude that there must exist smooth functions ${}^M D_{AB} \in C^\infty(M,\mathbb{R})$ and (local) constants ${}^Q D_{AB}$ on Q such that

$${}^M\Delta_{AB} \oplus {}^Q\Delta_{AB} = \mathsf{d}^{(0)}_{\Delta_Q}\big({}^M D_{AB} \oplus {}^Q D_{AB}\big)\,,$$

which reproduces Eq. (5.32).

We now obtain

$$[\![\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]\!]^{(\mathrm{H},\omega;\Delta_Q)} = f_{ABC}\,\widetilde{\mathfrak{K}}_C + 0 \oplus \big(\mathsf{d}({}^M D_{AB} - \mathrm{c}_{(AB)}), -\Delta_Q({}^M D_{AB} - \mathrm{c}_{(AB)}) - {}^Q D_{AB}\big)$$

and, in the notation of the proof of Eq. (5.29),

$$\big[\,\widetilde{\mathfrak{K}}_{A\,\mathrm{i}}\,,\,\widetilde{\mathfrak{K}}_{B\,\mathrm{i}}\,\big]_{\mathrm{V}} = \mathrm{e}^{-\mathrm{pr}^*_{\mathsf{L}_{Q|(\pi,\varepsilon)}}{}^M E_{(\pi,\varepsilon)\,\mathrm{i}}} \triangleright \widetilde{\mathsf{L}}^{Q|(\pi,\varepsilon)}_{(1,1\sqcup 0)}[\![\,\mathfrak{K}_A\,,\,\mathfrak{K}_B\,]\!]^{(\mathrm{H},\omega;\Delta_Q)} = f_{ABC}\,\widetilde{\mathfrak{K}}_{C\,\mathrm{i}}$$

which, indeed, yields Eqs. (5.33) and (5.34). This concludes the proof.

8. The gauge anomaly – the sixfold way

In the preceding sections, we have identified a specific target-space model of the algebraic structure on the set of charges of a rigid symmetry of the multi-phase σ-model and elucidated the underlying simple and universal differential-geometric/cohomological scheme that is realised both in the presence as well as in the absence of (symmetry-preserving) world-sheet defects. In the course of our study, we have laid considerable emphasis on the very fundamental gerbe-theoretic aspects of the said structure, or – to put it differently – on the naturalness of that structure in the setting of a target-space geometry with the 2-category of bundle gerbes with connection over it. This leaves us with a fairly complete understanding of infinitesimal rigid symmetries of the two-dimensional field theory of interest.

In this last section, we want to take our analysis of σ-model symmetries to the next level by considering a local variant thereof. A prerequisite for an in-depth treatment of the subject is a precise identification and systematisation of potential obstructions to rendering a global symmetry of the σ-model local. This task was completed in a series of papers [GR03, Gaw05, SSW07, GSW08, GSW11, GSW10, GSW12] in which a cohomological classification scheme was worked out for these so-called gauge anomalies and from which a universal Gauge Principle has emerged.

In the intrinsically geometric context of the σ-model, the field-theoretic gauging procedure admits a clear-cut interpretation: It boils down to extending the target space by a principal G_σ-bundle over the world-sheet and subsequently coupling the string background to the attendant principal G_σ-connection 1-form in a manner that allows to descend the thus extended string background, with its metric and gerbe-theoretic structure, to the coset of the original target space with respect to the action of the symmetry group G_σ. This yields the σ-model on the coset of the original target space by the action of the group whenever the latter coset exists within the smooth category. Accordingly, gauge anomalies quantify obstructions to the existence of equivalences between a given string background and the one obtained through pullback of a string background from the coset.

The rationale for taking up the issue of the gauge anomaly here is twofold: First of all, we want to understand how gauge anomalies are encoded in the algebroidal bracket structure introduced earlier, and – in so doing – reassess the naturalness of the latter in the context of the study of σ-model symmetries. As the bracket captures infinitesimal features of the symmetries, we anticipate to gain insights into the so-called small gauge anomaly in this manner. The relation, established previously, between the Poisson algebra of Noether charges of the rigid symmetry on the phase space of the σ-model and the (relative) twisted Courant algebroid of the corresponding σ-symmetric sections of the restricted generalised tangent sheaves over the target space gives rise to an additional expectation, to wit, that there is a canonical (*i.e.* symplectic) interpretation of the small gauge anomaly. This expectation will receive confirmation in the framework of a canonical description of the gauged σ-model of Ref. [GSW12, Sect. 10.2] that we develop hereunder along the lines of Ref. [Sus11, Sect. 3]. The second piece of motivation for studying gauge anomalies derives from the findings of Ref. [Sus11] that establish an intimate relation between dualities of the σ-model (including the geometric symmetries studied in the present paper) and a distinguished class of conformal defects. Thus, we shall demonstrate how gauge anomalies, both large and small, obstruct the existence of topological defect quivers implementing the action of the gauge group $C^\infty(\Sigma,\mathrm{G}_\sigma)$ on states of the gauged σ-model in the manner detailed in Ref. [Sus11, Sec. 4]. As argued in Refs. [JK06, Sec. 4],[RS09, Sec. 2.9] and [FFRS09, Sect. 3], defect quivers of this kind give rise to a much intuitive world-sheet definition of the coset σ-model. The definition, taking as the point of departure the original string background (existing before the action of the symmetry group G_σ has been divided out), identifies embeddings of the world-sheet in that

background related by the action of the gauge group and admits embeddings that are continuously differentiable *up to the action of the gauge group*, the latter being realised by means of an arbitrarily fine mesh of homotopically deformable (at no cost in the action functional) $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect lines, with defect junctions that can be pulled through one another (once again, at no cost in the action functional). Our construction of a full-fledged duality background for the gauged σ-model in the presence of defects transparent to the symmetries gauged will be seen to give substance to some general claims of Ref. [Sus11, Sec. 4] concerning the duality-defect correspondence, and – at the same time – will provide an explicit realisation of the somewhat abstract Duality Scheme laid out in Ref. [Sus11, Rem. 5.6]. Clearly, the fundamental concept that interrelates the various facets of the gauge anomaly outlined above is the gerbe theory of the σ-model that serves to characterise the anomaly itself, underlies the structure of the Courant algebroid, and – finally – (co-)determines the world-sheet definition of the coset σ-model.

By way of preparation for the subsequent reinterpretations of the gauge anomaly, let us recapitulate the relevant definitions and results from Ref. [GSW12]. We begin with

Convention 8.1. In order to unclutter the notation, we fix a convention for pullbacks. Thus, for any p-form η on a smooth space $\mathscr{M} := \mathscr{M}_1 \times \mathscr{M}_2 \times \cdots \times \mathscr{M}_N$ equipped with canonical projections $\mathrm{pr}_{i_1,i_2,\ldots,i_n} : \mathscr{M} \to \mathscr{M}_{i_1} \times \mathscr{M}_{i_2} \times \cdots \times \mathscr{M}_{i_n}$ given for $1 \le i_1 < i_2 < \ldots < i_n \le N$, we denote

$$\eta_{[i_1,i_2,\ldots,i_n]^*} := \mathrm{pr}^*_{i_1,i_2,\ldots,i_n}\eta\,.$$

In particular,

$$\eta_{i^*} \equiv \eta_{[i]^*} = \mathrm{pr}_i^*\eta$$

for any $1 \le i \le N$. Analogous convention will be used for geometric objects such as bundles, gerbes *etc.*

Given a triple of smooth manifolds $\mathscr{M}_1,\mathscr{M}_2,\mathscr{N}$ and a pair of smooth maps $f_i : \mathscr{M}_i \to \mathscr{N}$, denote by

$$\mathscr{M}_1{}_{f_1}\!\times_{f_2}\mathscr{M}_2 := \{\ (m_1,m_2) \in \mathscr{M}_1 \times \mathscr{M}_2 \quad | \quad f_1(m_1) = f_2(m_2)\ \}$$

the product of the $\mathscr{M}_i$ fibred over $\mathscr{N}$.

✓

We may now state

Definition 8.2. [GSW12, Cor. 3.17] Adopt the notation of Definition 3.2 and of Corollary 7.5. The **gauged two-dimensional non-linear σ-model for network-field configurations $(X\,|\,\Gamma)$ in string background $\mathfrak{B}$ on world-sheet (Σ,γ) with defect quiver Γ** coupled to a topologically trivial gauge field $\mathrm{A} \in \Omega^1(\Sigma) \otimes \mathfrak{g}_\sigma$ is a theory of continuously differentiable maps $X\ :\ \Sigma \to \mathscr{F}$ determined by the principle of least action applied to the action functional

$$S_\sigma[(X\,|\,\Gamma);\mathrm{A},\gamma] := -\tfrac{1}{2}\int_\Sigma \mathrm{g}_\mathrm{A}(\mathsf{d}\xi \overset{\wedge}{,} \star_\gamma \mathsf{d}\xi) - \mathsf{i}\log\mathrm{Hol}_{\mathcal{G}_\mathrm{A},\Phi_\mathrm{A},(\varphi_{n\,\mathrm{A}})}(\xi\,|\,\Gamma)\,, \tag{8.1}$$

where $\xi = (\mathrm{id}_\Sigma, X)\ :\ \Sigma \to \Sigma \times \mathscr{F}$ is the extended embedding field, and the extended string background $\mathfrak{B}_\mathrm{A} := (\mathcal{M}_\mathrm{A},\mathcal{B}_\mathrm{A},\mathcal{J}_\mathrm{A})$ defining the action functional has the following components:

- the extended target $\mathcal{M}_\mathrm{A}$ composed of the target space $\Sigma \times M$ with the metric

$$\mathrm{g}_\mathrm{A} := \mathrm{g}_{2^*} - \mathrm{K}_{A\,2^*} \otimes \mathrm{A}^A_{1^*} - \mathrm{A}^A_{1^*} \otimes \mathrm{K}_{A\,2^*} + \mathrm{h}_{AB\,2^*}\left(\mathrm{A}^A \otimes \mathrm{A}^B\right)_{1^*}\,, \tag{8.2}$$

  written in terms of the tensors

$$\mathrm{K}_A := \mathrm{g}({}^M\mathscr{K}_A,\cdot)\,, \qquad\qquad \mathrm{h}_{AB} := \mathrm{g}({}^M\mathscr{K}_A,{}^M\mathscr{K}_B)$$

  and implicitly understood to act on the second tensor factor in

$$\mathsf{d}\xi(\sigma) = \left(\mathsf{d}\sigma^a \otimes \partial_a, \partial_a X^\mu(\sigma)\,\mathsf{d}\sigma^a \otimes \partial_\mu|_{X(\sigma)}\right)\,,$$

  and with the gerbe

$$\mathcal{G}_\mathrm{A} = \mathcal{G}_{2^*} \otimes I_{\rho_\mathrm{A}}\,, \tag{8.3}$$

  containing in its definition a trivial gerbe I_{ρ_A} with a global curving

$$\rho_\mathrm{A} := \kappa_{A\,2^*} \wedge \mathrm{A}^A_{1^*} - \tfrac{1}{2}\,\mathrm{c}_{AB\,2^*}\left(\mathrm{A}^A \wedge \mathrm{A}^B\right)_{1^*} \in \Omega^2(\Sigma \times M)\,; \tag{8.4}$$

- the extended bi-brane $\mathcal{B}_{\rm A}$ with the world-volume $\Gamma\times Q$, bi-brane maps $\underline{\iota}_\alpha = {\rm id}_\Gamma\times\iota_\alpha,\ \alpha\in\{1,2\}$, the curvature
$$\omega_{\rm A} = \omega_{2^*} - \underline{\Delta}_Q\rho_{\rm A} + {\mathsf d}\lambda_{\rm A}\,, \tag{8.5}$$
written in terms of the pullback operator $\underline{\Delta}_Q := \underline{\iota}_2^* - \underline{\iota}_1^*$ and
$$\lambda_{\rm A} := -k_{A\,2^*}\,{\rm A}^A_{1^*} \in \Omega^1(\Sigma\times Q)\,, \tag{8.6}$$
and with the 1-isomorphism
$$\Phi_{\rm A} = \Phi_{2^*}\otimes J_{\lambda_{\rm A}}\,, \tag{8.7}$$
written in terms of a trivial 1-isomorphism (a trivial bundle) $J_{\lambda_{\rm A}}$ with a global connection 1-form $\lambda_{\rm A}$;
- the extended inter-bi-brane $\mathcal{J}_{\rm A}$ with component world-volumes $\mathfrak{V}_\Gamma^{(n)}\times T_n,\ n\geq 3$, defined in terms of the subsets $\mathfrak{V}_\Gamma^{(n)}\subset\mathfrak{V}_\Gamma$ composed of vertices of valence n, with inter-bi-brane maps $\underline{\pi}_n^{k,k+1} = {\rm id}_{\mathfrak{V}_\Gamma^{(n)}}\times\pi_n^{k,k+1},\ k\in\overline{1,n}$, and with 2-isomorphisms $\varphi_{n\,{\rm A}} := \varphi_{n\,2^*}$.

Fundamental invariance properties of the gauged σ-model are expressed in the following two theorems in which we also define the notions of the small and large gauge anomaly used in the remainder of the paper.

Theorem 8.3. [GSW10, Prop. 3.1][GSW12, Cor. 3.11] *In the notation of Definition 8.2 and of Corollary 7.5, the action functional of Eq.* (8.1) *is invariant under infinitesimal gauge transformations*
$$\begin{aligned} X^\mu(\sigma) &\mapsto X^\mu(\sigma) + \Lambda^A(\sigma)\,{}^{\mathscr{F}}\mathscr{K}_A^\mu\,(X(\sigma))\,, \\ {\rm A}_a^A(\sigma) &\mapsto {\rm A}_a^A(\sigma) + \partial_a\Lambda^A(\sigma) - f_{ABC}\,\Lambda^B(\sigma)\,{\rm A}_a^C(\sigma)\,, \end{aligned} \tag{8.8}$$
written in terms of arbitrary functions $\Lambda^A\in C^\infty(\Sigma,\mathbb{R})$, iff the following ***conditions for a consistent gauging*** *are satisfied:*
$$\mathscr{L}_{\mathscr{K}_A}\kappa_B = f_{ABC}\,\kappa_C\,,\qquad \mathscr{L}_{\mathscr{K}_A}k_B = f_{ABC}\,k_C\,,\qquad {\rm c}_{(AB)} = 0\,. \tag{8.9}$$

Definition 8.4. In the notation of Corollary 7.5, the **small gauge anomaly** of the σ-model (8.1) is the obstruction to the existence of a choice of the objects $(\kappa_A,k_A),\ A\in\overline{1,\dim\mathfrak{g}_\sigma}$ satisfying relations (8.9).

The large gauge anomaly is most neatly described in the language of Lie groupoids whose theory was developed in Refs. [Mac87, MM03]. Below, we set up our notation by way of preparation for the discussion to follow.

Definition 8.5. A **groupoid** is the septuple ${\rm Gr} = ({\rm Ob\,Gr},{\rm Mor\,Gr},s,t,{\rm Id},{\rm Inv},\circ)$ composed of a pair of sets: the **object set** ${\rm Ob\,Gr}$ and the **arrow set** ${\rm Mor\,Gr}$, and a quintuple of **structure maps**: the **source map** $s:{\rm Mor\,Gr}\to{\rm Ob\,Gr}$ and the **target map** $t:{\rm Mor\,Gr}\to{\rm Ob\,Gr}$, the **unit map** ${\rm Id}:{\rm Ob\,Gr}\to{\rm Mor\,Gr}: m\mapsto{\rm Id}_m$, the **inverse map** ${\rm Inv}:{\rm Mor\,Gr}\to{\rm Mor\,Gr}:\overrightarrow{g}\mapsto\overrightarrow{g}^{-1}\equiv{\rm Inv}(\overrightarrow{g})$, and the **multiplication map** $\circ:{\rm Mor\,Gr}\,{}_s\!\times_t{\rm Mor\,Gr}\to{\rm Mor\,Gr}:(\overrightarrow{g},\overrightarrow{h})\mapsto\overrightarrow{g}\circ\overrightarrow{h}$. The structure maps satisfy the consistency conditions (whenever the expressions are well-defined):

(i) $s(\overrightarrow{g}\circ\overrightarrow{h}) = s(\overrightarrow{h}),\ t(\overrightarrow{g}\circ\overrightarrow{h}) = t(\overrightarrow{g})$;
(ii) $(\overrightarrow{g}\circ\overrightarrow{h})\circ\overrightarrow{k} = \overrightarrow{g}\circ(\overrightarrow{h}\circ\overrightarrow{k})$;
(iii) ${\rm Id}_{t(\overrightarrow{g})}\circ\overrightarrow{g} = \overrightarrow{g} = \overrightarrow{g}\circ{\rm Id}_{s(\overrightarrow{g})}$;
(iv) $s(\overrightarrow{g}^{-1}) = t(\overrightarrow{g}),\ t(\overrightarrow{g}^{-1}) = s(\overrightarrow{g}),\ \overrightarrow{g}\circ\overrightarrow{g}^{-1} = {\rm Id}_{t(\overrightarrow{g})},\ \overrightarrow{g}^{-1}\circ\overrightarrow{g} = {\rm Id}_{s(\overrightarrow{g})}$.

Thus, a groupoid is a (small) category with all morphisms invertible.

A **morphism** between two groupoids ${\rm Gr}_i,\ i\in\{1,2\}$ is a functor $\Phi:{\rm Gr}_1\to{\rm Gr}_2$.

A **Lie groupoid** is a groupoid whose object and arrow sets are smooth manifolds, whose structure maps are smooth, and whose source and target maps are surjective submersions. A morphism between two Lie groupoids is a functor between them with smooth object and morphism components.

We may now proceed towards an analysis of geometric symmetries of the σ-model to be gauged.

Definition 8.6. Let $\mathscr{M}$ be a smooth manifold, and let G be a group. A **left action of group** G **on manifold** $\mathscr{M}$ is a smooth map

$$ {}^{\mathscr{M}}\ell \;:\; \mathrm{G}\times\mathscr{M}\to\mathscr{M} \;:\; (g,m)\mapsto g.m\equiv{}^{\mathscr{M}}\ell_g(m)\,. $$

A manifold equipped with a left action of group G is termed a **G-space**.

The **action groupoid** associated to a G-space $\mathscr{M}$ is a Lie groupoid, usually denoted as

$$ \mathrm{G}\ltimes\mathscr{M} \;:\; \mathrm{G}\times\mathscr{M} \overset{s}{\underset{t}{\rightrightarrows}} \mathscr{M}\;, $$

with the object and morphism sets

$$ \mathrm{Ob}\,(\mathrm{G}\ltimes\mathscr{M})=\mathscr{M}\,,\qquad\qquad \mathrm{Mor}\,(\mathrm{G}\ltimes\mathscr{M})=\mathrm{G}\times\mathscr{M}\,, $$

with the source and target maps

$$ s(g,x):=x\,,\qquad\qquad t(g,x):=g.x\,, $$

with the identity morphisms

$$ \mathrm{Id}_x:=(e,x) $$

(e is the group unit), with the inversion map

$$ \mathrm{Inv}(g,m):=\big(g^{-1},g.m\big)\equiv(g,m)^{-1}\,, $$

and, finally, with the composition of morphisms

$$ (g,h.x)\circ(h,x):=(g\cdot h,x)\,. $$

The nerve of this category, termed the **nerve of action groupoid** $\mathrm{G}\ltimes\mathscr{M}$ and denoted as

$$ \mathsf{N}^\bullet(\mathrm{G}\ltimes\mathscr{M}) \quad:\quad \cdots \substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow} \mathrm{G}^2\times\mathscr{M} \substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow} \mathrm{G}\times\mathscr{M} \rightrightarrows \mathscr{M}\;, \tag{8.10} $$

is an incomplete simplicial object in the category of G-spaces equipped with **face maps**

$$ {}^{\mathscr{M}}d_i^{(m)} \;:\; \mathrm{G}^m\times\mathscr{M}\to\mathrm{G}^{m-1}\times\mathscr{M}\,,\qquad i\in\overline{0,m-1} $$

explicitly given by

$$ \begin{aligned} {}^{\mathscr{M}}d_0^{(m)}(g_m,g_{m-1},\ldots,g_1,x) &= (g_{m-1},g_{m-2},\ldots,g_1,x)\,,\\ {}^{\mathscr{M}}d_m^{(m)}(g_m,g_{m-1},\ldots,g_1,x) &= (g_m,g_{m-1},\ldots,g_2,g_1.x)\,,\\ {}^{\mathscr{M}}d_i^{(m)}(g_m,g_{m-1},\ldots,g_1,x) &= (g_m,g_{m-1},\ldots,g_{m+2-i},g_{m+1-i}\cdot g_{m-i},g_{m-1-i},\ldots,g_1,x)\,. \end{aligned} $$

It is over the nerve of the action groupoid $\mathrm{G}_\sigma\ltimes\mathscr{F}$ that the construction of the gauged σ-model is carried out. We start with the topologically trivial case.

Proposition 8.7. [GSW12, Thm. 4.12] *Adopt the notation of Definitions 3.2, 8.2 and 8.6, of Corollary 7.5, and of Example 6.6, and denote by* G_σ *the symmetry group of the σ-model of Definition 3.2 susceptible of gauging, with the Lie algebra* $\mathfrak{g}_\sigma$*. The action functional of Eq.* (8.1) *is invariant under gauge transformations*

$$ X(\sigma)\mapsto{}^{\mathscr{F}}\ell\,(\chi(\sigma),X(\sigma))\equiv\chi.X(\sigma)\,,\qquad\qquad \mathrm{A}(\sigma)\mapsto\mathrm{Ad}_{\chi(\sigma)}\mathrm{A}(\sigma)-\mathsf{d}\chi\,\chi^{-1}(\sigma)\equiv{}^\chi\mathrm{A}(\sigma)\,, $$

written in terms of an arbitrary function $\chi\in C^\infty(\Sigma,\mathrm{G}_\sigma)$*, iff there exist: a 1-isomorphism*

$$ \Upsilon \;:\; {}^Md_1^{(1)\,*}\mathcal{G}\xrightarrow{\cong}{}^Md_0^{(1)\,*}\mathcal{G}\otimes I_\rho \tag{8.11} $$

of gerbes over $\mathrm{G}_\sigma\times M$*, with*

$$ \rho:=\kappa_{A\,2^*}\wedge\theta^A_{L\,1^*}-\tfrac{1}{2}\,\mathrm{c}_{AB\,2^*}\,\big(\theta^A_L\wedge\theta^B_L\big)_{1^*}\in\Omega^2(\mathrm{G}_\sigma\times M)\,, $$

and a 2-isomorphism

$$ \Xi \;:\; {}^Qd_1^{(1)\,*}\Phi\xRightarrow{\cong}\big(\big(\iota_2^{(1)\,*}\Upsilon^{-1}\otimes\mathrm{id}\big)\circ\big({}^Qd_0^{(1)\,*}\Phi\otimes\mathrm{id}\big)\circ\iota_1^{(1)\,*}\Upsilon\big)\otimes J_\lambda \tag{8.12} $$

between the 1-isomorphisms over $\mathrm{G}_\sigma\times Q$*, with* $\iota_\alpha^{(1)}:=\mathrm{id}_{\mathrm{G}_\sigma}\times\iota_\alpha$ *and*

$$ \lambda:=-k_{A\,2^*}\,\theta^A_{L\,1^*}\in\Omega^1(\mathrm{G}_\sigma\times Q)\,, $$

such that the identities

$$ {}^{T_n}d_1^{(1)\,*}\varphi_n=\psi_1\bullet\big(\mathrm{id}\circ{}^{T_n}d_0^{(1)\,*}\varphi_n\circ\mathrm{id}\big)\bullet\psi_2\bullet\big(\big(\Xi_n^{n,1\,(1)}\otimes\mathrm{id}\big)\circ\big(\Xi_n^{n-1,n\,(1)}\otimes\mathrm{id}\big)\circ\cdots\circ\Xi_n^{1,2\,(1)}\big) \tag{8.13} $$

hold true over $\mathrm{G}_\sigma \times T_n$ *(for all* $n \geq 3$*) for the 2-isomorphisms* $\Xi_n^{k,k+1\,(1)} = \left(\mathrm{id}_{\mathrm{G}_\sigma} \times \pi_n^{k,k+1}\right)^* \Xi^{\varepsilon_n^{k,k+1}}$ *and for certain 2-isomorphisms* $\psi_\beta,\ \beta \in \{1,2\}$ *canonically determined by components of* $\mathfrak{B}$ *and by* Υ*.*

Physical considerations presented in Ref. [GSW12] seem to imply that a consistent quantum field theory of the gauged σ-model requires incorporating topologically non-trivial gauge fields into the lagrangean description. These are represented by principal G_σ-connection 1-forms on arbitrary principal G_σ-bundles over Σ, for which our conventions are summarised in

Definition 8.8. Let G be a (topological) group, and Σ a topological space[14]. A **principal** G**-bundle over base** Σ is the quadruple $\mathcal{P} := (\mathsf{P}, \Sigma, \pi_{\mathsf{P}}, r_{\mathsf{P}})$ composed of a fibre bundle $\pi_{\mathsf{P}} : \mathsf{P} \to \Sigma$ with total space P and base Σ, and of a continuous free and transitive fibrewise right action

$$r_{\mathsf{P}} \ :\ \mathsf{P} \times \mathrm{G} \to \mathsf{P} \ :\ (p,g) \mapsto r(p,g) \equiv p.g\,.$$

Under a local trivialisation

$$\tau_i \ :\ \pi_{\mathsf{P}}^{-1}(\Sigma_i) \to \Sigma_i \times \mathrm{G}$$

associated with a choice $\{\Sigma_i\}_{i \in \mathscr{I}}$ of an open cover of Σ and defining a local section $\sigma_i : \Sigma_i \to \pi_{\mathsf{P}}^{-1}(\Sigma_i)$ in the standard manner,

$$\sigma_i(\sigma) := \tau_i^{-1}(\sigma, e)\,,$$

the above action is related to the action of G on itself by right regular translations,

$$\tau_i^{-1}(\sigma, g).h = \tau_i^{-1}(\sigma, g \cdot h)\,.$$

For G a Lie group with Lie algebra $\mathfrak{g}$, the latter having generators $t_A,\ A \in \overline{1, \dim \mathfrak{g}}$ subject to the structure relations

$$[t_A, t_B] = f_{ABC}\, t_C \tag{8.14}$$

written in terms of structure constants f_{ABC}, a **connection (1-form) on** $\mathcal{P}$, also termed the **principal** G**-connection 1-form**, is a $\mathfrak{g}$-valued 1-form $\mathcal{A} \in \Omega^1(\mathsf{P}) \otimes \mathfrak{g}$ with the defining properties

$$\mathcal{A}(p.g) = \mathrm{Ad}_{g^{-1}} \mathcal{A}(p)\,, \qquad\qquad {}^{\mathsf{P}}\mathscr{K}_A \lrcorner \mathcal{A} = t_A\,,$$

expressed in terms of the **fundamental vector fields** ${}^{\mathsf{P}}\mathscr{K}_A$ on P determined by the formula

$$({}^{\mathsf{P}}\mathscr{K}_A f)(p) := \tfrac{\mathsf{d}}{\mathsf{d}t}\big|_{t=0} f\left(\mathrm{e}^{-t t_A}.p\right)\,,$$

valid for any $f \in C^\infty(\mathsf{P}, \mathbb{R})$.

Finally, let $(\mathscr{M}, {}^{\mathscr{M}}\ell)$ be a (topological) G-space in the sense of Definition 8.5. The **bundle associated to** $\mathcal{P}$ **by (left) action** ${}^{\mathscr{M}}\ell$, also termed the **associated bundle** for short whenever there is no risk of confusion, is the fibre bundle

$$\pi_{\mathsf{P} \times_{\mathrm{G}} \mathscr{M}} \ :\ \mathsf{P} \times_{\mathrm{G}} \mathscr{M} \equiv (\mathsf{P} \times \mathscr{M})/\mathrm{G} \to \Sigma$$

obtained as the smooth quotient of the product bundle $\mathsf{P} \times \mathscr{M} \to \Sigma$ with respect to the right action $\widetilde{r}$ of G on the total space $\mathsf{P} \times \mathscr{M}$ given by

$$\widetilde{r} \ :\ (\mathsf{P} \times \mathscr{M}) \times \mathrm{G} \to \mathsf{P} \times \mathscr{M} \ :\ \left((p,m), g\right) \mapsto \left(r_{\mathsf{P}}(p,g), {}^{\mathscr{M}}\ell(g^{-1}, m)\right), \tag{8.15}$$

and equipped with the projection

$$\pi_{\mathsf{P} \times_{\mathrm{G}} \mathscr{M}}\left([(p,m)]\right) := \pi_{\mathsf{P}}(p)\,.$$

We may now formulate

Definition 8.9. [GSW12, Def. 10.1] Adopt the notation of Definitions 3.2 and 8.2, and let $\pi_{\mathsf{P}} : \mathsf{P} \to \Sigma$ be a principal G_σ-bundle over Σ with a principal connection 1-form $\mathcal{A} \in \Omega^1(\mathsf{P}) \otimes \mathfrak{g}_\sigma$. A P**-extended string background** is the string background $\widetilde{\mathfrak{B}}_{\mathcal{A}} := (\widetilde{\mathcal{M}}_{\mathcal{A}}, \widetilde{\mathcal{B}}_{\mathcal{A}}, \widetilde{\mathcal{J}}_{\mathcal{A}})$ with the following components

- the P**-extended target** $\widetilde{\mathcal{M}}_{\mathcal{A}}$ composed of the target space $\widetilde{M} = \mathsf{P} \times M$ with the metric $\widetilde{\mathrm{g}}_{\mathcal{A}}$ and the gerbe $\widetilde{\mathcal{G}}_{\mathcal{A}}$ defined as in Eqs. (8.2) and (8.3), respectively, but with the global connection 1-form A on Σ replaced by $\mathcal{A}$;
- the P**-extended** $\widetilde{\mathcal{G}}_{\mathcal{A}}$**-bi-brane** $\widetilde{\mathcal{B}}_{\mathcal{A}}$ with the world-volume $\widetilde{Q} = \mathsf{P}|_\Gamma \times Q$, with the bi-brane maps $\widetilde{\iota}_\alpha = \mathrm{id}_{\mathsf{P}} \times \iota_\alpha,\ \alpha \in \{1,2\}$, and with the curvature $\widetilde{\omega}_{\mathcal{A}}$ and the 1-isomorphism $\widetilde{\Phi}_{\mathcal{A}}$ defined as in Eqs. (8.5) and (8.7), respectively, but with the global connection 1-form A on Σ replaced by $\mathcal{A}$;

[14] In the context of the present paper, the definition will usually be restricted to the smooth category.

- the P**-extended** $(\widetilde{\mathcal{G}}_{\mathcal{A}}, \widetilde{\mathcal{B}}_{\mathcal{A}})$**-inter-bi-brane** $\widetilde{\mathcal{J}}_{\mathcal{A}}$ with component world-volumes $\widetilde{T}_n = \mathsf{P}|_{\mathfrak{V}_\Gamma^{(n)}} \times T_n$, $n \geq 3$, with inter-bi-brane maps $\widetilde{\pi}_n^{k,k+1} = \mathrm{id}_{\mathsf{P}} \times \pi_n^{k,k+1}$, $k \in \mathbb{Z}/n\mathbb{Z}$, and 2-isomorphisms $\widetilde{\varphi}_{n\,\mathcal{A}} = \varphi_{n\,2^*}$.

The idea behind the introduction of the P-extended string background is that it permits to write the coupling between the original string background and the topologically non-trivial gauge field in a manner that generalises the treatment of the topologically trivial case. This comes at a price: The target space $\widetilde{\mathscr{F}} = \mathsf{P} \times \mathscr{F}$ of the P-extended string background is not the physical space of the corresponding gauged σ-model. In order to keep the original field content, we have to pass to the (smooth) quotient $\widetilde{\mathscr{F}}/\mathrm{G}_\sigma \cong \mathscr{F}$ with respect to the combined (right) action

$$
{}^{\widetilde{\mathscr{F}}}\ell \;:\; \widetilde{\mathscr{F}} \times \mathrm{G}_\sigma \to \widetilde{\mathscr{F}} \;:\; \big((p,x),g\big) \mapsto \big(r_{\mathsf{P}}(p,g), {}^{\mathscr{F}}\ell\big(g^{-1},x\big)\big)\,.
$$

We arrive thereby at associated bundles. Dividing out ${}^{\widetilde{\mathscr{F}}}\ell$ is straightforward on the level of the target space, and the true challenge is to ensure that also the geometric structure supported by $\widetilde{\mathscr{F}}$ descends to the quotient space in the sense rendered rigorous in Ref. [GSW12, Sec. 8]. To describe such circumstances, we need

Definition 8.10. [GSW12, Def. 8.7] Adopt the notation of Definitions 3.2 and 8.6, of Corollary 7.5, and of Proposition 8.7. Let $\{\tau^A\}^{A\in\overline{1,\dim\mathfrak{g}}}$ be the generators of $\mathfrak{g}_\sigma^*$ dual to the generators $\{t_A\}_{A\in\overline{1,\dim\mathfrak{g}}}$ of $\mathfrak{g}_\sigma$ satisfying Eq. (8.14). A $(\mathrm{G}_\sigma,\rho,\lambda)$**-equivariant string background** is a triple $\mathfrak{B}_{(\mathrm{G}_\sigma,\rho,\lambda)} := (\mathcal{M}_{(\mathrm{G}_\sigma,\rho)}, \mathcal{B}_{(\mathrm{G}_\sigma,\lambda)}, \mathcal{J}_{\mathrm{G}_\sigma})$ with the following components:

- a (G_σ,ρ)**-equivariant target** $\mathcal{M}_{(\mathrm{G}_\sigma,\rho)} := (M, \mathrm{g}, (\mathcal{G},\Upsilon,\gamma))$, consisting of a target space M carrying the structure of a G_σ-space with a G_σ-invariant metric g, and of a gerbe $\mathcal{G}$ with a G_σ-invariant curvature H admitting a $\mathfrak{g}_\sigma$-equivariantly closed G_σ-equivariant (Cartan-model) extension $\widehat{\mathrm{H}} = \mathrm{H} - \kappa$, $\kappa = \kappa_A \otimes \tau^A$, and endowed with a (G_σ,ρ)-equivariant structure, *i.e.* coming with a 1-isomorphism Υ of gerbes over $\mathrm{G}_\sigma \times M$ as in Eq. (8.11) and with a 2-isomorphism

$$
\begin{tikzcd}
\big({}^M d_1^{(1)} \circ {}^M d_1^{(2)}\big)^* \mathcal{G} \arrow[r, "{}^M d_2^{(2)\,*}\Upsilon"] \arrow[d, "{}^M d_1^{(2)\,*}\Upsilon"'] & \big({}^M d_1^{(1)} \circ {}^M d_0^{(2)}\big)^* \mathcal{G} \otimes I_{{}^M d_2^{(2)\,*}\rho} \arrow[d, "{}^M d_0^{(2)\,*}\Upsilon \otimes \mathrm{id}"] \arrow[dl, Rightarrow, "\gamma" description] \\
\big({}^M d_0^{(1)} \circ {}^M d_1^{(2)}\big)^* \mathcal{G} \otimes I_{{}^M d_1^{(2)\,*}\rho} \arrow[r, equal] & \big({}^M d_0^{(1)} \circ {}^M d_0^{(2)}\big)^* \mathcal{G} \otimes I_{{}^M d_0^{(2)\,*}\rho + {}^M d_2^{(2)\,*}\rho}
\end{tikzcd}
\tag{8.16}
$$

  between the 1-isomorphisms over $\mathrm{G}_\sigma^2 \times M$, satisfying, over $\mathrm{G}_\sigma^3 \times M$, the coherence condition

$$
{}^M d_1^{(3)\,*}\gamma \bullet \big(\mathrm{id} \circ {}^M d_3^{(3)\,*}\gamma\big) = {}^M d_2^{(3)\,*}\gamma \bullet \big(\big({}^M d_0^{(3)\,*}\gamma \otimes \mathrm{id}\big) \circ \mathrm{id}\big)\,; \tag{8.17}
$$

- a $(\mathrm{G}_\sigma,\lambda)$**-equivariant** $\mathcal{G}$**-bi-brane** $\mathcal{B}_{(\mathrm{G}_\sigma,\lambda)} = (\mathcal{B},\Xi)$, consisting of a bi-brane $\mathcal{B}$ with a world-volume Q carrying the structure of a G_σ-space, with a G_σ-invariant curvature ω admitting a G_σ-equivariant (Cartan-model) extension $\widehat{\omega} = \omega - k$, $k = k_A \otimes \tau^A$ satisfying the relations

$$
\widehat{\mathsf{d}}\widehat{\omega} = -\Delta_Q \widehat{\mathrm{H}}\,, \qquad\qquad \Delta_{T_n}\widehat{\omega} = 0\,,
$$

  and endowed with a $(\mathrm{G}_\sigma,\lambda)$-structure, *i.e.* coming with a 2-isomorphism Ξ over $\mathrm{G}_\sigma \times Q$ as in Eq. (8.12), subject to the coherence condition

$$
\big(\big(\iota_2^{(1)\,*}\gamma^\sharp \otimes \mathrm{id}\big) \circ \mathrm{id}\big) \bullet {}^Q d_1^{(2)\,*}\Xi = \big(\mathrm{id} \circ \iota_1^{(1)\,*}\gamma\big) \bullet \big(\mathrm{id} \circ \big({}^Q d_0^{(2)\,*}\Xi \otimes \mathrm{id}\big) \circ \mathrm{id}\big) \bullet {}^Q d_2^{(2)\,*}\Xi \tag{8.18}
$$

  imposed over $\mathrm{G}_\sigma^2 \times Q$;
- a G_σ**-equivariant** $(\mathcal{G},\mathcal{B})$**-inter-bi-brane** $\mathcal{J}_{\mathrm{G}_\sigma} = \mathcal{J}$ defined by a $(\mathcal{G},\mathcal{B})$-inter-bi-brane $\mathcal{J}$ with component world-volumes T_n each carrying the structure of a G_σ-space, and with 2-isomorphisms φ_n satisfying relation (8.13).

Remark 8.11. Above, the Cartan model of $\mathfrak{g}_\sigma$-equivariant cohomology of the G_σ-space $\mathscr{F}$ is taken with the $\mathfrak{g}_\sigma$-equivariant differential defined on $\mathfrak{g}_\sigma$-equivariant p-forms η, polynomially dependent on elements of $\mathfrak{g}_\sigma(\ni V)$, by the expression

$$
\widehat{\mathsf{d}}\eta(V) = \mathsf{d}\eta(V) - \overline{V} \lrcorner\, \eta(V)\,,
$$

where $\overline{V}$ is the vector field acting on smooth functions on $\mathscr{F}$ according to the formula

$$(\overline{V}f)(m) := \tfrac{\mathsf{d}}{\mathsf{d}t}\big|_{t=0} f\left(\mathrm{e}^{-tV}.m\right) . \tag{8.19}$$

The imposition of the requirements that both H and ω admit $\mathfrak{g}_\sigma$-equivariant extensions, and that the latter compose a (Δ_Q, Δ_{T_n})-relative $\mathfrak{g}$-equivariant 3-cocycle guarantees that the small gauge anomaly vanishes.

Furthermore, $\circ$ and $\bullet$ are – respectively – the horizontal and the vertical composition of 1- and 2-isomorphisms of the 2-category of bundle gerbes with connection (over the relevant base $\mathrm{G}_\sigma^m \times \mathscr{F}$), *cf.* Ref. [Wal07, Sect. 1.2], and $\psi^\sharp : \Psi_2^{-1} \xrightarrow{\cong} \Psi_1^{-1}$ is a 2-isomorphism canonically induced by a given 2-isomorphism $\psi : \Psi_1 \xrightarrow{\cong} \Psi_2$ in a manner detailed in Ref. [Wal07, Sect. 1.3].

We may now phrase the important

Theorem 8.12. [GSW12, Thm. 9.7, Cor. 10.9 & Def. 10.10] *In the notation of Definitions 3.2 and 8.9, and of Proposition 8.7, a* P*-extended string background* $\widetilde{\mathfrak{B}}_{\mathcal{A}}$ *with target space* $\widetilde{\mathscr{F}}$ *descends to a unique string background* $\mathfrak{B}_{\mathcal{A}}$ *with target space* $\widetilde{\mathscr{F}}/\mathrm{G}_\sigma$ *if the underlying string background* $\mathfrak{B}$ *carries a* $(\mathrm{G}_\sigma, \rho, \lambda)$*-equivariant structure of Definition 8.10. The descendant string background* $\mathfrak{B}_{\mathcal{A}}$ *then defines the gauged* σ*-model coupled to a gauge field* $\mathcal{A}$*.*

The last theorem motivates

Definition 8.13. In the notation of Proposition 8.7 and of Definition 8.10, the **large gauge anomaly** of the gauged σ-model is the obstruction to the existence of a choice of a 1-isomorphism Υ and 2-isomorphisms γ and Ξ satisfying relations (8.17), (8.18) and (8.13).

Remark 8.14. A comment is due on the status of the gauging procedure outlined, and – consequently – also on that of the gauge anomaly. It ought to be kept in mind that the procedure involves choices, such as, *e.g.*, the choice of the coupling between the string background and the gauge field (at most quadratic in the latter, and in this sense 'minimal') determining the form of the small gauge anomaly, and that even for this distinguished choice of the coupling the existence of a G_σ-equivariant structure on the string background, tantamount to the vanishing of the large gauge anomaly, is a *sufficient* condition for a consistent gauging of the global symmetry, and not an obviously *necessary* one (excepting the case of discrete symmetries, treated in all generality in Ref. [GSW11], to which the present procedure applies through reduction, and in which the answer is known to be unique). It is, therefore, imperative to back up our proposal for the universal gauge principle with additional structural evidence attesting its naturalness and versatility as a tool of description of the σ-model with a local symmetry. Steps towards this end were taken already in Ref. [GSW12] where the small gauge anomaly was given a simple interpretation in the framework of $\mathfrak{g}_\sigma$-equivariant cohomology of the target space $\mathscr{F}$ (Sec. 3.2, *ib.*, but *cf.* also Refs. [JJMO90, HS89, FOS94, FOS, Wit92, Wu93, GSW10] for earlier results in this direction), where an infinitesimal analogon of the G_σ-equivariant structure was extracted from a local analysis of the string background with a vanishing small gauge anomaly (Sec. 7, *ib.*), and where the necessity of the existence of a full-fledged G_σ-equivariant structure was demonstrated in the special situation in which the coset target space $\mathscr{F}/\mathrm{G}_\sigma$ is a smooth manifold (Sec. 9, *ib.*). Below, we take up anew the pursuit of evidence in favour of the proposal of Ref. [GSW12], putting it in the context of the canonical description of the σ-model in the presence of conformal defects, with emphasis on their relation to σ-model dualities, and that of the generalised geometry of rigid symmetries of the two-dimensional field theory of interest.

8.1. **The canonical description of the gauged σ-model.** In this first part of our discussion, we shall set up a canonical description of the gauge anomaly. From this description, the vanishing of the anomaly will be seen to emerge as a sufficient condition for the existence of a hamiltonian realisation of the Lie algebra $\mathfrak{g}_\sigma$ of the symmetry group G_σ on the phase space of the σ-model, consistent with the splitting-joining interactions and admitting a canonical extension to a realisation of $C^\infty(\Sigma, \mathfrak{g}_\sigma)$ as a gauge-symmetry algebra on the phase space of the gauged σ-model coupled 'minimally' to a topologically trivial gauge field. Here, the term "gauge symmetry" indicates, in keeping with, *e.g.*, Ref. [Gaw72], that the vector fields generating infinitesimal gauge transformations on functions on the said phase space belong to the characteristic distribution of the presymplectic form of the gauged σ-model.

We commence by introducing the main elements of the subsequent analysis.

Definition 8.15. Adopt the notation of Definitions 3.2, 8.2, I.2.6, I.3.5, I.3.9 and I.3.10, and that of Propositions 3.5 and I.3.8. Let $\pi_{\mathcal{F}_\sigma} : \mathcal{F}_\sigma \to \Sigma$ be the covariant configuration bundles of the non-linear σ-model of Definition 3.2. The **covariant configuration bundles of the** corresponding **gauged non-linear σ-model** of Definition 8.2 are given by the fibred product

$$\pi_{\mathcal{F}_\sigma} \circ \mathrm{pr}_1 \ : \ \widetilde{\mathcal{F}}_\sigma \coloneqq \mathcal{F}_\sigma\, {}_{\pi_{\mathcal{F}_\sigma}}\!\times_{\pi_{\mathsf{T}^*\Sigma}} (\mathsf{T}^*\Sigma \otimes \mathfrak{g}_\sigma) \to \Sigma \,.$$

For the associated first-jet bundles, $\mathsf{J}^1\widetilde{\mathcal{F}}_\sigma \to \Sigma$, we shall use the set of local coordinates from Definition I.3.5, augmented by local coordinates $(\mathrm{A}^A_a, \zeta^A_{ab})$, $A \in \overline{1, \dim \mathfrak{g}_\sigma}$, $a, b \in \{1, 2\}$ on the fibre of $\mathsf{J}^1\mathsf{T}^*\Sigma \otimes \mathfrak{g}_\sigma$.

Similarly, we shall parameterise classical sections of $\mathsf{J}^1\widetilde{\mathcal{F}}_\sigma$ (understood in the sense of an obvious extension of Proposition I.3.8 to the setting of the gauged σ-model) by their Cauchy data localised on a suitable space-like contour $\mathscr{C} \subset \Sigma$. The **state space of the gauged non-linear σ-model** $\widetilde{\mathsf{P}}_\sigma \subset \Gamma\big(\mathsf{J}^1\widetilde{\mathcal{F}}_\sigma\big)$ composed of these classical sections splits naturally into the untwisted and N-twisted sectors (with $N \in \mathbb{N} \setminus \{0\}$), *cf.* Remark I.2.11, and the respective Cauchy data take the following form:

- in the untwisted sector, to be denoted as $\widetilde{\mathsf{P}}_{\sigma,\varnothing}$, they are given by a quadruple $(X, \mathsf{p}, \mathrm{A}, \Pi)$ composed of smooth loops $X : \mathbb{S}^1 \to M$ and $\mathrm{A} : \mathbb{S}^1 \to \Omega^1(\Sigma) \otimes \mathfrak{g}_\sigma$, and of the respective normal covector fields p and Π (*cf.* Definition I.3.9), where the latter pair, (A, Π), is implicitly determined by the former one, (X, p), through the Euler–Lagrange equations for A,

$$\begin{pmatrix} \mathrm{h}_{AB} & -\mathrm{c}_{[AB]} \\ -\mathrm{c}_{[AB]} & \mathrm{h}_{AB} \end{pmatrix} \begin{pmatrix} (\widehat{n} \lrcorner \mathrm{A}^B) \\ (\widehat{t} \lrcorner \mathrm{A}^B) \end{pmatrix} = \begin{pmatrix} \mathrm{K}_{A\,\mu} & \kappa_{A\,\mu} \\ \kappa_{A\,\mu} & \mathrm{K}_{A\,\mu} \end{pmatrix} \begin{pmatrix} (X_*\widehat{n})^\mu \\ (X_*\widehat{t})^\mu \end{pmatrix}, \qquad \mathrm{c}_{[AB]} \coloneqq \tfrac{1}{2}\,(\mathrm{c}_{AB} - \mathrm{c}_{BA})\,, \tag{8.20}$$

obtained by varying the action functional of Eq. (8.1) in the direction of the world-sheet gauge field[15], *cf.* Ref. [GSW12, Sec. 9], and written in terms of the vector field $\widehat{t}$ tangent to $\mathscr{C}$ and defining its orientation, and of the vector field $\widehat{n} = \eta^{-1}\big(\widehat{t} \lrcorner \mathrm{Vol}(\Sigma, \eta), \cdot\big)$ normal to it.
- in the N-twisted sector, to be denoted as $\widetilde{\mathsf{P}}_{\sigma,\mathcal{B}\,|\,(P_k,\varepsilon_k)}$, they are given by the $(2N+4)$-tuple $(X, \mathsf{p}, q_k, V_k, \mathrm{A}, \Pi \,|\, k \in \overline{1,N})$ composed of smooth maps $X : \mathbb{S}^1_{\{P_k\}} \to M$ and $\mathrm{A} : \mathbb{S}^1 \to \Omega^1(\Sigma) \otimes \mathfrak{g}_\sigma$ (a loop), of the respective normal covector fields p and Π, and of N points $(q_k, V_k) \in \mathsf{T}\underline{Q}$, related to (X, p) as in Definition I.3.10. Here, the submanifold

$$\underline{Q} \coloneqq \bigcap_{A=1}^{\dim \mathfrak{g}_\sigma} k_A^{-1}\big(\{0\}\big) \subset Q$$

is assumed smooth, *cf.* Ref. [GSW12, Eq. (9.5)], and the pair (A, Π) is determined by the pair (X, p) through Eq. (8.20) taken in conjunction with the condition of continuity of the gauge field along $\mathbb{S}^1_{\{P_k\}}$.

We readily establish

Proposition 8.16. *Adopt the notation of Proposition 3.3, of Corollary 7.5, and of Definitions 8.2 and 8.15. A (pre)symplectic form on* $\widetilde{\mathsf{P}}_{\sigma,\varnothing}$,

$$\widetilde{\Omega}_{\sigma,\varnothing}[(X, \mathsf{p}, \mathrm{A}, \Pi)] = \Omega_{\sigma,\varnothing}[(X, \mathrm{p})] + \delta \int_{\mathbb{S}^1} \mathrm{Vol}\big(\mathbb{S}^1\big) \wedge \Delta(X; \mathrm{A})\,, \tag{8.21}$$

and that on $\widetilde{\mathsf{P}}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$,

$$\widetilde{\Omega}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}[(X, \mathsf{p}, q, V, \mathrm{A}, \Pi)] = \Omega_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}[(X, \mathrm{p}, q, V)] + \delta \int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big) \wedge \Delta(X; \mathrm{A})\,, \tag{8.22}$$

differ from their counterparts from Proposition 3.3 by the 1-form

$$\Delta(X; \mathrm{A}) \coloneqq \big(\widehat{t} \lrcorner \mathrm{A}^A\big)\, X^*\kappa_A - \big(\widehat{n} \lrcorner \mathrm{A}^A\big)\, X^*\mathrm{K}_A\,.$$

Proof. An easy exercise using the basic methods of the first-order formalism recapitulated in Ref. [Sus11, Sec. 3]. □

We are now fully prepared to give a canonical interpretation of the small gauge anomaly. We begin with the unextended σ-model, prior to the gauging. Putting together Propositions 3.13, 4.5 and 5.11 and Theorems 6.3 and 6.5, we obtain

[15] Recall that we have fixed a minkowskian gauge for the world-sheet metric η.

Theorem 8.17. *If the small gauge anomaly of Theorem 8.3 vanishes, then there exists a hamiltonian realisation of the symmetry Lie algebra $\mathfrak{g}_\sigma$ of Proposition 3.5 on the full state space of the two-dimensional non-linear σ-model of Definition 3.2, and that realisation is continuous across the defect quiver of the σ-model in the sense of Proposition 4.5 and consistent with the splitting-joining interactions in the sense of Theorems 6.3 and 6.5.*

Proof. This is a simple corollary to the propositions and theorems listed above. □

The significance of the small gauge anomaly in the canonical description of the gauged σ-model is emphasised by the following

Theorem 8.18. *If the small gauge anomaly of Theorem 8.3 vanishes, then there exists a canonical extension of each vector field generating the action of the symmetry Lie algebra $\mathfrak{g}_\sigma$ of Proposition 3.5 on smooth functions on the full state space of the two-dimensional non-linear σ-model of Definition 3.2 to a vector field generating the action of $C^\infty(\Sigma,\mathfrak{g}_\sigma)$ as a gauge-symmetry algebra on the full state space of the gauged two-dimensional non-linear σ-model of Definition 8.2.*

Proof. In the proof, we focus on the 1-twisted case exclusively. Clearly, our result generalises straightforwardly to the N-twisted case with $N \geq 1$ arbitrary. Furthermore, the claim for the untwisted case can readily be recovered from what follows through a trivialisation of the twist, *i.e.* through setting $\omega = 0$, $\gamma_A = 0 = k_A$ and $\iota_1 = \mathrm{id}_M = \iota_2$.

Let $\Lambda^A t_A \in \mathfrak{g}_\sigma$ and $\overline{\Lambda} := \Lambda^A \mathscr{K}_A \in \Gamma(\mathsf{T}\mathscr{F})$, and write the corresponding vector field on $\mathsf{P}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$ explicitly as

$$\begin{aligned}\widetilde{\mathsf{L}}^{Q\,|\,(\pi,\varepsilon)}_{\iota_\alpha\,*}\overline{\Lambda}[(X,\mathsf{p},q,V)] \;&=\; \int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big)\,\Lambda^A\left[{}^M\mathscr{K}^\mu_A\,(X(\cdot))\,\tfrac{\delta}{\delta X^\mu(\cdot)} - \mathsf{p}_\nu(\cdot)\,\partial_\mu\,{}^M\mathscr{K}^\nu_A\big(X(\cdot)\big)\,\tfrac{\delta}{\delta \mathsf{p}_\mu(\cdot)}\right]\\ &\quad+\Lambda^A\,{}^Q\mathscr{K}^\mu_A(q)\,\tfrac{\delta}{\delta X^\mu(q)}\end{aligned}$$

Let, next, $\Lambda^A(\cdot)\,t_A \in C^\infty(\Sigma,\mathfrak{g}_\sigma)$, and abbreviate

$$\mathrm{A}^A_x := \widehat{x} \lrcorner\, \mathrm{A}^A\,,\qquad\qquad \Lambda^A_{,x} := \widehat{x}\big(\Lambda^A\big)\,,\qquad x \in \{t,n\}\,.$$

Taking into account the transformation properties of the canonical variables of the gauged σ-model under the infinitesimal gauge transformation $\Lambda^A(\cdot)\,t_A$, we may easily write out the unique extension of $\widetilde{\mathsf{L}}^{Q\,|\,(\pi,\varepsilon)}_{\iota_\alpha\,*}\overline{\Lambda}$ in the form

$$\begin{aligned}&\widetilde{\Lambda}[(X,\mathsf{p},q,V,\mathrm{A},\Pi)]\\ :=\;& \int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big)\,\Big\{\Lambda^A(\cdot)\,{}^M\mathscr{K}^\mu_A\,(X(\cdot))\,\tfrac{\delta}{\delta X^\mu(\cdot)} - \big[\Lambda^A\,\mathsf{p}_\nu(\cdot)\,\partial_\mu\,{}^M\mathscr{K}^\nu_A\big(X(\cdot)\big) - \Lambda^A_{,n}(\cdot)\,\mathrm{K}_{A\,\mu}\,(X(\cdot))\big]\,\tfrac{\delta}{\delta \mathsf{p}_\mu(\cdot)}\\ &+\big(\partial_a\Lambda^A - f_{ABC}\,\Lambda^B\,\mathrm{A}^C_a\big)(\cdot)\,\tfrac{\delta}{\delta \mathrm{A}^A_a(\cdot)}\Big\} + \Lambda^A(\pi)\,{}^Q\mathscr{K}^\mu_A(q)\,\tfrac{\delta}{\delta X^\mu(q)}\,.\end{aligned}$$

It now remains to check that $\widetilde{\Lambda}$ is in the kernel of $\widetilde{\Omega}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}$. Upon invoking the defining formulæ for the $\kappa_A, \mathrm{c}_{AB}$ and h_{AB}, we obtain

$$\begin{aligned}&\widetilde{\Lambda} \lrcorner\, \widetilde{\Omega}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}[(X,\mathsf{p},q,V,\mathrm{A},\Pi)]\\ =\;& \int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big)\,\Big\{-\Lambda^A(\cdot)\big[{}^M\mathscr{K}^\mu_A\,(X(\cdot))\,\delta\mathsf{p}_\mu(\cdot) - X_*\widehat{t}(\cdot)\lrcorner\,\delta\kappa_A\,(X(\cdot))\\ &+\mathrm{A}^B_n(\cdot)\,(\mathscr{L}_{\mathscr{K}_A}\mathrm{K}_B - \delta\mathrm{h}_{AB})\,(X(\cdot)) - \mathrm{A}^B_t(\cdot)\,(\mathscr{L}_{\mathscr{K}_A}\kappa_B - \delta\mathrm{c}_{AB})\,(X(\cdot))\\ &+\mathrm{c}_{AB}\,(X(\cdot))\,\delta\mathrm{A}^B_t(\cdot) - \mathrm{h}_{AB}\,(X(\cdot))\,\delta\mathrm{A}^B_n(\cdot) + \mathsf{p}_\mu(\cdot)\,\delta^M\mathscr{K}^\mu_A\,(X(\cdot))\big] + \Lambda^A_{,n}(\cdot)\,\mathrm{K}_A\,(X(\cdot))\\ &+\big(\Lambda^A_{,t} - f_{ABC}\,\Lambda^B\,\mathrm{A}^C_t\big)(\cdot)\,\kappa_A\,(X(\cdot)) - \big(\Lambda^A_{,n} - f_{ABC}\,\Lambda^B\,\mathrm{A}^C_n\big)(\cdot)\,\mathrm{K}_A\,(X(\cdot))\Big\}\\ &+\varepsilon\,\Lambda^A(\pi)\,{}^Q\mathscr{K}_A \lrcorner\,\omega(q)\,.\end{aligned}$$

Further simplification of the last expression is achieved with the help of the Killing equations for the ${}^M\mathscr{K}_A$ and the defining formulæ for the k_A. Altogether, we have

$$\begin{aligned}&\widetilde{\Lambda} \lrcorner\, \widetilde{\Omega}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}[(X,\mathsf{p},q,V,\mathrm{A},\Pi)]\\ =\;& \int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big)\,\big[-\Lambda^A(\cdot)\,\delta\widetilde{J}_A(X,\mathsf{p},\mathrm{A})(\cdot) + \Lambda^A\,\mathrm{A}^B_t(\cdot)\,\left(\mathscr{L}_{\mathscr{K}_A}\kappa_B - f_{ABC}\,\kappa_C\right)(X(\cdot))\big]\end{aligned}$$

$$-\varepsilon\,\Lambda^A(\pi)\,\delta k_A(q)\,,$$

where

$$\widetilde{J}_A(X,\mathrm{p};\mathrm{A}) \coloneqq {}^M\mathscr{K}_A(X) \lrcorner\, \mathrm{p} + X_*\widehat{t} \lrcorner\, \kappa_A(X) - \mathrm{h}_{AB}(X)\,\mathrm{A}^B_n + \mathrm{c}_{AB}(X)\,\mathrm{A}^B_t$$

are extensions of the Noether currents $J_{\mathfrak{K}_A}$ of Eq. (3.26).

At this stage, we may start using the field equations[16] for the gauge field alongside conditions (8.9), whereby the above immediately reduces to the form

$$\widetilde{\Lambda} \lrcorner\, \widetilde{\Omega}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}[(X,\mathsf{p},q,V,\mathrm{A},\Pi)] = -\int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big)\,\Lambda^A(\cdot)\,\delta\widetilde{J}_A(X,\mathsf{p},\mathrm{A})(\cdot)\,.$$

Taking into account the explicit formula

$$\mathsf{p} = \mathrm{g}_{\mu\nu}(X)\,(X_*\widehat{n})^\mu\,\delta X^\nu\,,$$

we ultimately arrive at the expression

$$\widetilde{\Lambda} \lrcorner\, \widetilde{\Omega}_{\sigma,\mathcal{B}\,|\,(\pi,\varepsilon)}[(X,\mathsf{p},q,V,\mathrm{A},\Pi)] = -\delta\int_{\mathbb{S}^1_{\{\pi\}}} \mathrm{Vol}\big(\mathbb{S}^1_{\{\pi\}}\big)\,\mathrm{c}_{(AB)}\,(X(\cdot))\,\Lambda^A\,\mathrm{A}^B_t(\cdot)$$

which vanishes identically whenever the small gauge anomaly does. □

In the present section, we have identified the small gauge anomaly as an obstruction to the existence of a canonical realisation of the infinitesimal gauge symmetry of the gauged σ-model. This is to be viewed as an alternative derivation of the corresponding results obtained in the lagrangean picture in Ref. [GSW12]. Taking guidance from the intuition developed in Ref. [Sus11], we are next led to expect that the vanishing of the large gauge anomaly ensures, in turn, the existence of a lift of the integrated (*i.e.* finite) version of the symmetry to an automorphism of the pre-quantum bundle of the gauged σ-model. This expectation will be corroborated and – indeed – extended to a proper world-sheet definition of the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-reduction of the gauged σ-model in Section 8.3. In the meantime, we pause to give a very natural and purely geometric interpretation of the small gauge anomaly, (apparently) in abstraction from the underlying two-dimensional field theory.

8.2. **The generalised-geometric/groupoidal interpretation.** Below, we want to reexamine the small gauge anomaly from an altogether different, intrinsically geometric perspective offered by the previously introduced algebroidal structure on the set of σ-symmetric sections of the restricted tangent sheaves over the target space of the σ-model, *cf.* Corollary 7.5. One of the key results of this section was already announced in Ref. [GSW12, Sec. 6]. However, in view of its relevance to a more complete understanding of rigid symmetries of the σ-model, as well as of the obvious structural connection to the rest of the present paper, we have decided to restate it in the language of Section 7.2.2, in this manner avoiding unnecessary repetitions. Thus, we shall reinterpret conditions (8.9) for a consistent gauging by establishing a straightforward link between the $(\mathrm{H}\oplus\omega\oplus 0)$-twisted (Δ_Q,Δ_{T_n})-relative Courant bracket on $\mathfrak{E}_{(\iota_\alpha,\pi_n^{k,k+1})}(\mathscr{F}) \cong \Gamma_{(\iota_\alpha,\pi_n^{k,k+1})}\big(\widehat{\mathsf{E}}^{(1,2\sqcup 1\sqcup 0)}\mathscr{F}\big)$ and the action groupoid $\mathrm{G}_\sigma{\ltimes}\mathscr{F}$. The latter emerges from the discussion of large gauge transformations and the ensuing construction of a G_σ-equivariant string background, and so its appearance in the analysis of the gauge anomaly is not very surprising. That it is actually quite natural will be demonstrated in the second part of the present section in which we establish a correspondence between the data of the gauged σ-model and the category of principal bundles over Σ with the structural groupoid $\mathrm{G}_\sigma{\ltimes}\mathscr{F}$. We shall elaborate the correspondence in Section 8.3, where it will be shown to bridge the gap between the infinitesimal description of the gauge symmetry and its finite counterpart.

In what follows, we shall make frequent use of basic notions and constructions of the theory of Lie groupoids and Lie algebroids, and so we assume working knowledge thereof on the reader's part. For an in-depth treatment of the theory, consult Refs. [Mac87, MM03].

In order to set the stage for subsequent considerations, we recall

Definition 8.19. Let $\mathscr{M}$ be a smooth manifold. A **Lie algebroid** over the **base** $\mathscr{M}$ is a quintuple $\mathfrak{Gr} = \big(V,\mathscr{M},[\cdot,\cdot],\pi_V,\alpha_{\mathsf{T}\mathscr{M}}\big)$ composed of a vector bundle $\pi_V : V \to \mathscr{M}$, a Lie bracket $[\cdot,\cdot]$ on the vector space $\Gamma(V)$ of its sections, and a bundle map $\alpha_{\mathsf{T}\mathscr{M}} : V \to \mathsf{T}\mathscr{M}$ termed the **anchor (map)**. These are required to have the following properties:

[16] Note that the last of them, $k_A = 0$, implies the middle one of equalities (8.9).

(i) the induced map $\Gamma(\alpha_{\mathsf{T}\mathscr{M}}):\Gamma(V)\to\Gamma(\mathsf{T}\mathscr{M})$, to be denoted by the same symbol $\alpha_{\mathsf{T}\mathscr{M}}$ in what follows, is a Lie-algebra homomorphism (with respect to the standard Lie-algebra structure on $\Gamma(\mathsf{T}\mathscr{M})$ defined by the Lie bracket of vector fields);

(ii) $[\cdot,\cdot]$ obeys the **Leibniz identity**

$$[X,f\,Y]=f\,[X,Y]+\alpha_{\mathsf{T}\mathscr{M}}(X)(f)\,Y$$

for all $X,Y\in\Gamma(V)$ and any $f\in C^\infty(M,\mathbb{R})$.

A **morphism** between two Lie algebroids $\mathfrak{Gr}_i=\big(V_i,\mathscr{M},[\cdot,\cdot]_i,\pi_{V_i},\alpha_{\mathsf{T}\mathscr{M}\,i}\big)$, $i\in\{1,2\}$ is a bundle map $\phi:V_1\to V_2$ that satisfies the relations[17]

$$\alpha_{\mathsf{T}\mathscr{M}\,1}=\alpha_{\mathsf{T}\mathscr{M}\,2}\circ\phi\,,\qquad\qquad \phi\circ[\cdot,\cdot]_1=[\cdot,\cdot]_2\circ(\phi\times\phi)\,.$$

We are now in a position to transcribe and quantify in the setting in hand the old observation (*cf.* Ref. [Gua03]) that a Courant bracket does not, in general, respect the Leibniz rule or the Jacobi identity, and so the associated Courant algebroid is *not* a Lie algebroid. The relevant objects are introduced in the following

Definition 8.20. Let $\mathscr{M}$ be a smooth manifold, $\pi_{\mathsf{E}}:\mathsf{E}\to\mathscr{M}$ a vector bundle equipped with a bundle map $\alpha_{\mathsf{T}\mathscr{M}}:\mathsf{E}\to\mathsf{T}\mathscr{M}$, an antisymmetric bracket $[\cdot,\cdot]_{\mathrm{C}}$ and a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the set of smooth sections of E. Assume that the quintuple $(\mathsf{E},\mathscr{M},[\cdot,\cdot]_{\mathrm{C}},\pi_{\mathsf{E}},\alpha_{\mathsf{T}\mathscr{M}})=:\mathfrak{C}$ satisfies the axioms of a Courant algebroid with base $\mathscr{M}$, as stated, *e.g.*, in Ref. [LWX98, Def. 2.1]. Take arbitrary $\mathfrak{X},\mathfrak{Y},\mathfrak{Z}\in\Gamma(\mathsf{E})$ and $f\in C^\infty(\mathscr{M},\mathbb{R})$, and endow $\mathfrak{C}$ with the natural structure of a $C^\infty(\mathscr{M},\mathbb{R})$-module (with respect to point-wise multiplication). The **Leibniz anomaly of** $\mathfrak{C}$ is defined as

$$\mathscr{L}\;:\;\Gamma(\mathsf{E})^2\times C^\infty(\mathscr{M},\mathbb{R})\to\Gamma(\mathsf{E})\;:\;(\mathfrak{X},\mathfrak{Y},f)\mapsto[\mathfrak{X},f\cdot\mathfrak{Y}]_{\mathrm{C}}-f\cdot[\mathfrak{X},\mathfrak{Y}]_{\mathrm{C}}-\big(\mathscr{L}_{\alpha_{\mathsf{T}\mathscr{M}}(\mathfrak{X})}f\big)\cdot\mathfrak{Y}$$

and the **Jacobi anomaly** (or **Jacobiator**) **of** $\mathfrak{C}$ is given by the formula

$$\mathscr{J}\;:\;\Gamma(\mathsf{E})^3\to\Gamma(\mathsf{E})\;:\;(\mathfrak{X},\mathfrak{Y},\mathfrak{Z})\mapsto[[\mathfrak{X},\mathfrak{Y}]_{\mathrm{C}},\mathfrak{Z}]_{\mathrm{C}}+[[\mathfrak{Z},\mathfrak{X}]_{\mathrm{C}},\mathfrak{Y}]_{\mathrm{C}}+[[\mathfrak{Y},\mathfrak{Z}]_{\mathrm{C}},\mathfrak{X}]_{\mathrm{C}}\,.$$

The two anomalies determine the obstruction to $\mathfrak{C}$ becoming a Lie algebroid.

Upon specialisation of the above general definition to the (relative-geometric) setting of interest, we establish

Theorem 8.21. [GSW12, Prop. 6.2] *Adopt the notation of Corollary 7.5, of Theorem 7.19, and of Definitions 3.2, 7.2, 7.11, 7.15, 7.18 and 8.20. Write*

$$K_A:=\kappa_A\oplus k_A\oplus0\in\Omega^1_{\mathrm{dR}}\left(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n}\right)\,,\qquad\qquad C_{AB}:=\mathrm{c}_{AB}\oplus0\in\Omega^0_{\mathrm{dR}}\left(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n}\right)\,,$$

and

$$\eta:=\mathrm{H}\oplus\omega\oplus0\in\Omega^3_{\mathrm{dR}}\left(M,Q,T_n\,|\,\Delta_Q,\Delta_{T_n}\right)\,.$$

In the basis $\{\Psi(\mathfrak{K}_A)=:\Psi_A\}_{A\in\overline{1,\dim\mathfrak{g}_\sigma}}$*, the obstruction to the involutivity of the* η*-twisted* (Δ_Q,Δ_{T_n})*-relative Courant bracket on* $\mathfrak{E}_{(\iota_\alpha,\pi_n^{k,k+1})}(\mathscr{F})$ *is given by the* (Δ_Q,Δ_{T_n})*-relative 1-cycle*

$$\alpha_{AB}=\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{M_{\mathscr{K}_A}}K_B-f_{ABC}\,K_C-\mathsf{d}^{(0)}_{(\Delta_Q,\Delta_{T_n})}C_{(AB)}\,,$$

and the Leibniz and Jacobi anomalies are determined by the expressions

$$\mathscr{L}(\Psi_A,\Psi_B,f)=-0\oplus C_{(AB)}\cdot\mathsf{d}^{(0)}_{(\Delta_Q,\Delta_{T_n})}(f,0)\,,$$

and

$$\begin{aligned}\mathscr{J}(\Psi_A,\Psi_B,\Psi_C)\;&=\;\big(f_{ABD}\,\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_C}+f_{CAD}\,\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_B}+f_{BCD}\,\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_A}\big)\,\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_D}\eta\\&\quad+\tfrac12\,\mathsf{d}^{(0)}_{(\Delta_Q,\Delta_{T_n})}\big(\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{M_{\mathscr{K}_A}}C_{[BC]}+\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{M_{\mathscr{K}_C}}C_{[AB]}+\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{M_{\mathscr{K}_B}}C_{[CA]}\big)\,,\end{aligned}$$

respectively.

Consequently, the triple

$$\mathfrak{S}_{\mathfrak{B}}:=\big(\mathsf{T}_{\mathfrak{E}}\mathscr{F},[\cdot,\cdot]^\eta_{\mathrm{C}},\alpha_{\mathsf{T}\mathscr{F}}\big)\,,$$

with $\mathsf{T}_{\mathfrak{E}}\mathscr{F}\subset\mathsf{T}\mathscr{F}$ *the subbundle whose space of sections is defined as the* $C^\infty(\mathscr{F},\mathbb{R})$*-linear span*

$$\Gamma(\mathsf{T}_{\mathfrak{E}}\mathscr{F}):=\oplus_{A=1}^{\dim\mathfrak{g}_\sigma}C^\infty(\mathscr{F},\mathbb{R})\,\Psi_A\,,$$

[17] Again, we use the same symbol for the bundle map and the induced map between spaces of sections.

with the obvious bundle projection $\pi_{\mathscr{F}}$, and with the $C^\infty(\mathscr{F},\mathbb{R})$-linear map $\alpha_{\mathsf{T}\mathscr{F}}$ defined on the base of $\Gamma(\mathsf{T}_{\mathfrak{E}}\mathscr{F})$ as

$$\alpha_{\mathsf{T}\mathscr{F}}(\Psi_A) = {}^{\mathscr{F}}\mathscr{K}_A\,,$$

carries a canonical structure of a Lie algebroid over $\mathscr{F}$ iff the small gauge anomaly, expressed in terms of the $\mathfrak{K}_A$, vanishes. The ensuing Lie algebroid is called the ***gauge-symmetry Lie algebroid of string background*** $\mathfrak{B}$*.*

Proof. Obvious, through inspection. □

Remark 8.22. It deserves to be emphasised that the intrinsic ambiguity in the definition of the σ-symmetric sections $\mathfrak{K}_A$ leaves room for nullifying the small gauge anomaly (and, consequently, for the application of the last part of the above theorem) even if the latter does not vanish for the original choice of the $\mathfrak{K}_A$. Indeed, consider two sets $\{\mathscr{K}_A \oplus K^i_A\}_{A\in\overline{1,\dim\mathfrak{g}_\sigma}}$, $K^i_A := \kappa^i_A \oplus k^i_A \oplus 0$, $i \in \{1,2\}$ of elements of $\mathfrak{E}_{(\iota_\alpha,\pi_n^{k,k+1})}(\mathscr{F})$ satisfying the defining relations (7.17). Write

$$\widetilde\Delta_A := K^2_A - K^1_A\,.$$

We find

$$\mathsf{d}^{(1)}_{(\Delta_Q,\Delta_{T_n})}\widetilde\Delta_A = 0\,, \tag{8.23}$$

and so the said ambiguity is parametrised by $\ker \mathsf{d}^{(1)}_{(\Delta_Q,\Delta_{T_n})}$. In fact, it is easy to identify those (Δ_Q,Δ_{T_n})-relative 1-cocycles whose contribution to the Courant bracket and to the scalar product is trivial in the sense that it cannot be used to cancel the anomalous terms α_{AB} and c_{AB} (obtained for the original sections $\mathscr{K}_A \oplus K^1_A$). To this end, we calculate, using Eq. (8.23),

$$\begin{aligned}
[\mathscr{K}_A \oplus (K^1_A + \widetilde\Delta_A), \mathscr{K}_B \oplus (K^1_B + \widetilde\Delta_B)] \;&=\; f_{ABC}\,\big(\mathscr{K}_C \oplus (K^1_C + \widetilde\Delta_C)\big) + 0 \oplus \alpha_{AB}\\
&\quad +0 \oplus \Big(\tfrac12\Big(\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_A}\widetilde\Delta_B - \mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_B}\widetilde\Delta_A\Big) - f_{ABC}\,\widetilde\Delta_C\Big)\,,
\end{aligned}$$

$$\big(\,\mathscr{K}_A \oplus (K^1_A + \widetilde\Delta_A)\,,\, \mathscr{K}_B \oplus (K^1_B + \widetilde\Delta_B)\,\big)_\lrcorner \;=\; \mathrm{c}_{(AB)} + \tfrac12\,\big(\imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_A}\widetilde\Delta_B + \imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_B}\widetilde\Delta_A\big)\,.$$

Thus, the conditions of triviality of the correction $\widetilde\Delta_A \in \ker \mathsf{d}^{(1)}_{(\Delta_Q,\Delta_{T_n})}$ read

$$\begin{cases} \tfrac12\Big(\mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_A}\widetilde\Delta_B - \mathscr{L}^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_B}\widetilde\Delta_A\Big) - f_{ABC}\,\widetilde\Delta_C = 0\,, \\[6pt] \imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_A}\widetilde\Delta_B + \imath^{(\Delta_Q,\Delta_{T_n})}_{\mathscr{K}_B}\widetilde\Delta_A = 0 \end{cases}\,,$$

or – equivalently –

$$\begin{cases} \mathscr{L}_{\mathscr{K}_A}\widetilde\Delta_B = f_{ABC}\,\widetilde\Delta_C \\[6pt] \big(\,\mathscr{K}_A \oplus \widetilde\Delta_A\,,\, \mathscr{K}_B \oplus \widetilde\Delta_B\,\big)_\lrcorner = 0 \end{cases}\,.$$

We conclude that the freedom in the choice of the K_A that can be employed to set the small gauge anomaly to zero is *effectively* parametrised by those (Δ_Q,Δ_{T_n})-relative 1-cocycles that do not define, upon tensoring with the $\tau^A \in \mathfrak{g}_\sigma^*$, their own (Δ_Q,Δ_{T_n})-relatively $\mathfrak{g}_\sigma$-equivariantly closed ($\mathfrak{g}_\sigma$-equivariant) extensions.

While the above result provides us with a neat quantitative description of the small gauge anomaly of the multi-phase σ-model, it leaves open questions concerning the nature of the ensuing Lie algebroid (in particular, its integrability to a Lie groupoid) and its intrinsic interpretation from the point of view of the geometry of the target space $\mathscr{F}$. An answer to the former question was given in Ref. [GSW12, Sec. 6], and we recall it below, only to set up the context for the analysis of the latter issue.

In order to be able to properly identify the gauge-symmetry Lie algebroid, we need to introduce additional formal tools.

Definition 8.23. Adopt the notation of Definitions 8.5 and 8.19. Denote by

$$R_{\overrightarrow{g}} \;:\; s^{-1}(\{t(\overrightarrow{g})\}) \to s^{-1}(\{s(\overrightarrow{g})\}) \;:\; \overrightarrow{h} \mapsto R_{\overrightarrow{g}}(\overrightarrow{h}) := \overrightarrow{h} \circ \overrightarrow{g}$$

the right multiplication map, written for an arbitrary $\overrightarrow{g} \in \mathrm{Mor\,Gr}$, and let $\mathfrak{X}^s_{\rm inv}(\mathrm{Mor\,Gr})$ be the vector space of **right Gr-invariant vector fields** on $\mathrm{Mor\,Gr}$, given by

$$\mathfrak{X}^s_{\rm inv}(\mathrm{Mor\,Gr}) = \{ \ \mathscr{V} \in \Gamma(\ker s_*) \quad | \quad R_{\cdot\,*}(\mathscr{V}) = \mathscr{V} \ \} .$$

The **tangent algebroid** of Gr is the Lie algebroid $\mathfrak{gr} = \big(\mathrm{Id}^*\ker s_*, \mathrm{Ob\,Gr}, [\cdot,\cdot], \pi_{\mathrm{Id}^*\ker s_*}, \alpha_{\mathsf{T}(\mathrm{Ob\,Gr})}\big)$ with (the obvious bundle projection $\pi_{\mathrm{Id}^*\ker s_*}$ and) the anchor $\alpha_{\mathsf{T}(\mathrm{Ob\,Gr})}$ inducing the map $t_* \circ i$ between spaces of sections, defined in terms of the canonical vector-bundle isomorphism

$$i \ : \ \mathfrak{X}^s_{\rm inv}(\mathrm{Mor\,Gr}) \xrightarrow{\cong} \Gamma(\mathrm{Id}^*\ker s_*) \,, \tag{8.24}$$

and with the Lie bracket given by the unique bracket on $\Gamma(\mathrm{Id}^*\ker s_*)$ for which i is an isomorphism of Lie algebras.

With hindsight, we next specialise the above definition to the case of $\mathrm{Gr} = \mathrm{G}\ltimes\mathscr{M}$, whereby we obtain

Proposition 8.24. *Adopt the notation of Definitions 8.5, 8.8 and 8.19, and of Example 6.6. The tangent algebroid of the action groupoid* $\mathrm{G}\ltimes\mathscr{M}$ *is the quintuple*

$$\mathfrak{g}\ltimes\mathscr{M} := (V, \mathscr{M}, [\cdot,\cdot]_{\mathfrak{g}\ltimes\mathscr{M}}, \pi_V, \alpha_{\mathsf{T}\mathscr{M}})$$

composed of

- *the vector bundle V with the space of sections*

$$\Gamma(V) := \bigoplus_{A=1}^{\dim\mathfrak{g}} C^\infty(\mathscr{M},\mathbb{R})\,\mathscr{R}_A$$

  spanned by vector fields

$$\mathscr{R}_A := i(R_A \circ \mathrm{pr}_1) \in \Gamma\left(\mathrm{Id}^*\ker \mathrm{pr}_{2\,*}\right)$$

  induced, through the isomorphism

$$i \ : \ \mathfrak{X}^{\mathrm{pr}_2}_{\rm inv}(\mathrm{G}\times\mathscr{M}) \xrightarrow{\cong} \Gamma\left(\mathrm{Id}^*\ker \mathrm{pr}_{2\,*}\right)$$

  of Eq. (8.24)*, from the right-invariant vector fields $R_A \circ \mathrm{pr}_1$ on $\mathrm{G}\times\mathscr{M}$, the latter being defined in terms of the standard right-invariant vector fields R_A on G dual to the right-invariant Maurer–Cartan 1-forms θ^A_R;*
- *the Lie bracket of smooth sections of V,*

$$[\lambda^A\,\mathscr{R}_A, \mu^B\,\mathscr{R}_B]_{\mathfrak{g}\ltimes\mathscr{M}} := f_{ABC}\,\lambda^A\,\mu^B\,\mathscr{R}_C + \left(\mathscr{L}_{\lambda^A\,{}^{\mathscr{M}}\mathscr{K}_A}\mu^B - \mathscr{L}_{\mu^A\,{}^{\mathscr{M}}\mathscr{K}_A}\lambda^B\right)\mathscr{R}_B \,,$$

  written, for arbitrary $\lambda^A, \mu^B \in C^\infty(\mathscr{M},\mathbb{R})$, in terms of the fundamental vector fields ${}^{\mathscr{M}}\mathscr{K}_A \equiv \bar{t}_A$ of Eq. (8.19)*;*
- *the $C^\infty(\mathscr{M},\mathbb{R})$-linear anchor map defined on the basis by the formula*

$$\alpha_{\mathsf{T}\mathscr{M}}(\mathscr{R}_A) := {}^{\mathscr{M}}\mathscr{K}_A \,.$$

The Lie algebroid thus defined is termed the ***action algebroid****.*

Proof. A constructive proof of the proposition, based directly on Definition 8.19, can be found in Ref. [GSW12, App. D]. □

We are now ready to state the important identification result.

Theorem 8.25. [GSW12, Thm. 6.1] *Adopt the notation of Definitions 3.2 and 7.18, of Corollary 7.5, of Proposition 8.24, and of Theorems 7.19 and 8.21. Whenever $\mathfrak{S}_{\mathfrak{B}}$ is a Lie algebroid, it is canonically isomorphic with the action algebroid $\mathfrak{g}_\sigma\ltimes\mathscr{F}$ in the sense of Definition 8.19.*

The emergence of the action groupoid $\mathrm{G}_\sigma\ltimes\mathscr{F}$ as the structure integrating those infinitesimal symmetries of the σ-model that can be consistently gauged harmonises nicely with its earlier appearance in the construction of a G_σ-equivariant string background, but, at the same time, it most definitely begs for elucidation, a task to which we turn next.

As mentioned earlier, physical consistency conditions appear to necessitate coupling the lagrangean fields of the σ-model with a target G_σ-space $\mathscr{F}$ to topologically non-trivial gauge fields. This is tantamount to

- introducing a (generic) principal G_σ-bundle $\mathsf{P}_{\mathrm{G}_\sigma}$ over the world-sheet, and subsequently
- replacing the σ-model field $X \in C^1(\Sigma,\mathscr{F})$ by a *global* section of the associated bundle $\mathsf{P}_{\mathrm{G}_\sigma}\times_{\mathrm{G}_\sigma}\mathscr{F}$.

It turns out that both constituents of the gauging algorithm listed above find a most natural interpretation in the theory of principal bundles with a structure Lie groupoid, as introduced (in the geometric form) in Ref. [Moe91] (*cf.* also Ref. [Hae84] for related work in the framework of the theory of foliations), developed in Ref. [MM03] and reviewed in Refs. [Ros04b, Ros04a] (from which we borrow some of the proofs and most of the notation) in a much accessible form in which the theory can be applied directly in the context in hand. It is the last observation that plays a central rôle in understanding the algebroidal interpretation of the small gauge anomaly, and – eventually – also in a reinterpretation of the large gauge anomaly. Therefore, with hindsight, we begin our discussion by introducing a few more formal tools and results.

Let us first set up the scene by introducing the concept of a principal bundle with a structure Lie groupoid. As it constitutes a categorification of the concept of a principal G-bundle from Definition 8.8, we start with

Definition 8.26. In the notation of Definition 8.5, a **right** Gr**-module space** is a triple $(\mathscr{M},\mu,\rho_{\mathscr{M}})$ composed of a smooth manifold $\mathscr{M}$, a smooth map $\mu:\mathscr{M}\to\mathrm{Ob}\,\mathrm{Gr}$ called the **momentum (of the action)**, and a smooth map

$$\rho_{\mathscr{M}}\ :\ \mathscr{M}\,{}_\mu\!\times_t\mathrm{Mor}\,\mathrm{Gr}\to\mathscr{M}\ :\ (m,\overrightarrow{g})\mapsto\rho_{\mathscr{M}}(m,\overrightarrow{g})\equiv m.\overrightarrow{g}$$

termed the **action (map)**. These satisfy the consistency conditions (whenever the expressions are well-defined):

(i) $\mu(m.\overrightarrow{g})=s(\overrightarrow{g})$;

(ii) $m.\mathrm{Id}_{\mu(m)}=m$;

(iii) $(m.\overrightarrow{g}).\overrightarrow{h}=m.(\overrightarrow{g}\circ\overrightarrow{h})$.

A left Gr-module space is defined similarly (with the rôles of the source and target maps in the definition interchanged).

The (right) action $\rho_{\mathscr{M}}$ is termed **free** iff the following implication obtains:

$$m.\overrightarrow{g}=m\qquad\Longrightarrow\qquad\overrightarrow{g}=\mathrm{Id}_{\mu(m)}\,,$$

so that, in particular, the **isotropy group** $\mathrm{Gr}_x=s^{-1}(\{x\})\cap t^{-1}(\{x\})$ of $x\in\mathrm{Ob}\,\mathrm{Gr}$ acts freely (in the usual sense) on the fibre $\mu^{-1}(\{x\})$.

The (right) action $\rho_{\mathscr{M}}$ is termed **transitive** iff for any two points $m,m'\in\mathscr{M}$ there exists an arrow $\overrightarrow{g}\in\mathrm{Mor}\,\mathrm{Gr}$ such that $m'=m.\overrightarrow{g}$.

Let $\mathrm{Gr}_i,\ i\in\{1,2\}$ be a pair of Lie groupoids and let $(\mathscr{M}_i,\mu_i,\rho_{\mathscr{M}_i})$ be the respective right-Gr_i-module spaces. A **morphism** between the latter is a pair (Θ,Φ) consisting of a smooth manifold map $\Theta:\mathscr{M}_1\to\mathscr{M}_2$ together with a functor $\Phi:\mathrm{Gr}_1\to\mathrm{Gr}_2$ for which the following diagrams commute

$$\begin{tikzcd}\mathscr{M}_1 \arrow[r, "\Theta"] \arrow[d, "\mu_1"'] & \mathscr{M}_2 \arrow[d, "\mu_2"] \\ \mathrm{Ob}\,\mathrm{Gr}_1 \arrow[r, "\Phi"] & \mathrm{Ob}\,\mathrm{Gr}_2\end{tikzcd}\ ,\tag{8.25}$$

$$\begin{tikzcd}\mathscr{M}_1{}_{\mu_1}\!\times_{t_1}\mathrm{Mor}\,\mathrm{Gr}_1 \arrow[r, "\Theta\times\Phi"] \arrow[d, "\rho_{\mathscr{M}_1}"'] & \mathscr{M}_2{}_{\mu_2}\!\times_{t_1}\mathrm{Mor}\,\mathrm{Gr}_2 \arrow[d, "\rho_{\mathscr{M}_2}"] \\ \mathscr{M}_1 \arrow[r, "\Theta"] & \mathscr{M}_2\end{tikzcd}\ .\tag{8.26}$$

We may now introduce the construct of immediate interest, to wit,

Definition 8.27. [Moe91, Sec. 1.2] In the notation of Definitions 8.5 and 8.26, a **principal** Gr**-bundle over base** $\mathscr{M}$ is a quintuple $\mathcal{P}=(\mathsf{P},\mathscr{M},\pi_{\mathsf{P}},\mu_{\mathsf{P}},\rho_{\mathsf{P}})$ composed of a pair of smooth manifolds: the **total space** P of the bundle and its **base** $\mathscr{M}$, and a triple of smooth maps: the surjective submersion $\pi_{\mathsf{P}}:\mathsf{P}\to\mathscr{M}$, termed the **bundle projection**, the **momentum (map)** $\mu:\mathsf{P}\to\mathrm{Ob}\,\mathrm{Gr}$, and the **action (map)** $\rho_{\mathsf{P}}:\mathsf{P}_\mu\!\times_t\mathrm{Mor}\,\mathrm{Gr}\to\mathsf{P}$ with the following properties:

(i) $(\mathsf{P},\mu_{\mathsf{P}},\rho_{\mathsf{P}})$ is a right Gr-module;

(ii) π_{P} is Gr-invariant in the sense made precise by the commutative diagram (in which pr_1 is the canonical projection)

$$\begin{tikzcd} \mathsf{P}\,{}_\mu\!\times_t \mathrm{Mor}\,\mathrm{Gr} \arrow[r, "\rho_{\mathsf{P}}"] \arrow[d, "\mathrm{pr}_1"'] & \mathsf{P} \arrow[d, "\pi_{\mathsf{P}}"] \\ \mathsf{P} \arrow[r, "\pi_{\mathsf{P}}"] & \mathscr{M} \end{tikzcd} ;$$

(iii) the map

$$(\mathrm{pr}_1, \rho_{\mathsf{P}}) \;:\; \mathsf{P}\,{}_\mu\!\times_t \mathrm{Mor}\,\mathrm{Gr} \to \mathsf{P}\,{}_{\pi_{\mathsf{P}}}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} \equiv \mathsf{P}^{[2]} \;:\; (p, \overrightarrow{g}) \mapsto (p, p.\overrightarrow{g})$$

is a diffeomorphism, so that Gr acts freely and transitively on π_{P}-fibres. The smooth inverse of $(\mathrm{pr}_1, \rho_{\mathsf{P}})$ takes the form

$$(\mathrm{pr}_1, \rho_{\mathsf{P}})^{-1} =: (\mathrm{pr}_1, \phi_{\mathsf{P}}), \qquad \phi_{\mathsf{P}} \;:\; \mathsf{P}^{[2]} \to \mathrm{Mor}\,\mathrm{Gr}$$

and ϕ_{P} is called the **division map**.

Let $(\mathsf{P}_i, \mathscr{M}, \pi_{\mathsf{P}_i}, \mu_i, \rho_{\mathsf{P}_i})$, $i \in \{1,2\}$ be a pair of principal Gr-bundles over a common base $\mathscr{M}$. A **morphism**[18] between the two bundles is a fibre-preserving morphism $(\Theta, \mathrm{Id}_{\mathrm{Gr}})$ between the corresponding right Gr-modules $(\mathsf{P}_i, \mu_i, \rho_{\mathsf{P}_i})$. The category of principal Gr-bundles over a smooth manifold $\mathscr{M}$ shall be denoted as $\mathrm{Gr}\text{-}\mathfrak{Bun}(\mathscr{M})$.

Some useful properties of the division map are summarised in the following

Proposition 8.28. [MM03, Sec. 5.7] *In the notation of Definitions 8.5 and 8.27, the division map ϕ_{P} of P has the following properties:*

(i) *it is determined uniquely by the relation*

$$q = p.\phi_{\mathsf{P}}(p, q),$$

valid for an arbitrary pair $(p, q) \in \mathsf{P}^{[2]}$;

(ii) $\phi_{\mathsf{P}}(p, q) \in \mathrm{Gr}_{\mu(q),\mu(p)}$, *where* $\mathrm{Gr}_{x,y} := s^{-1}(\{x\}) \cap t^{-1}(\{y\})$ *for any* $x, y \in \mathrm{Ob}\,\mathrm{Gr}$;

(iii) $\phi_{\mathsf{P}} \circ (\mathrm{Id}_{\mathsf{P}}, \mathrm{Id}_{\mathsf{P}}) = \mathrm{Id} \circ \mu$;

(iv) $\phi_{\mathsf{P}} \circ \tau = \mathrm{Inv} \circ \phi_{\mathsf{P}}$, *where* $\tau : \mathsf{P}^{[2]} \to \mathsf{P}^{[2]} : (p, q) \mapsto (q, p)$.

Proof. Obvious, though inspection. *Cf.* also Ref. [Ros04b, Sec. 4.3] for a simple proof. □

We have a counterpart of the well-known result for the category $\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma)$ of principal G-bundles over base Σ, to wit,

Proposition 8.29. [Moe91, Sec. 1.2] *In the notation of Definition 8.27, the category* $\mathrm{Gr}\text{-}\mathfrak{Bun}(\mathscr{M})$ *is a groupoid.*

Proof. Let $\mathcal{P}_i$, $i \in \{1,2\}$ be a pair of objects of $\mathrm{Gr}\text{-}\mathfrak{Bun}(\mathscr{M})$. Consider an arbitrary morphism $(\Theta, \mathrm{Id}_{\mathrm{Gr}}) \in \mathrm{Mor}_{\mathrm{Gr}\text{-}\mathfrak{Bun}(\mathscr{M})}(\mathcal{P}_1, \mathcal{P}_2)$. Assume that $p_1, p_2 \in \mathsf{P}_1$ satisfy the equation

$$\Theta(p_1) = \Theta(p_2), \tag{8.27}$$

whence also

$$\pi_{\mathsf{P}_1}(p_1) = \pi_{\mathsf{P}_2}\big(\Theta(p_1)\big) = \pi_{\mathsf{P}_2}\big(\Theta(p_2)\big) = \pi_{\mathsf{P}_1}(p_2) \qquad \Rightarrow \qquad (p_1, p_2) \in \mathsf{P}_1^{[2]}.$$

By property (iii) of Definition 8.27, and in virtue of point (i) of Proposition 8.28, we then have

$$p_2 = p_1.\phi_{\mathsf{P}_1}(p_1, p_2),$$

so that the Gr-equivariance of Θ, expressed by diagram (8.26), implies

$$\Theta(p_2) = \Theta(p_1).\phi_{\mathsf{P}_1}(p_1, p_2).$$

Taken in conjunction with the assumed equality (8.27), this yields

$$\begin{aligned} (\mathrm{pr}_1, \rho_{\mathsf{P}_2})\big(\Theta(p_1), \mathrm{Id}_{\mu_{\mathsf{P}_2}(\Theta(p_1))}\big) &= \big(\Theta(p_1), \Theta(p_1)\big) = \big(\Theta(p_1), \Theta(p_2)\big) = \big(\Theta(p_1), \Theta(p_1).\phi_{\mathsf{P}_1}(p_1, p_2)\big) \\ &= (\mathrm{pr}_1, \rho_{\mathsf{P}_2})\big(\Theta(p_1), \phi_{\mathsf{P}_1}(p_1, p_2)\big), \end{aligned}$$

[18] In Ref. [MM03, Sec. 5.7], these morphisms were termed "equivariant maps".

hence, owing to the invertibility of $(\mathrm{pr}_1, \rho_{\mathsf{P}_2})$,

$$\mathrm{Id}_{\mu_{\mathsf{P}_2}(\Theta(p_1))} = \phi_{\mathsf{P}_1}(p_1, p_2)\,.$$

Upon adducing the property of Θ encoded in diagram (8.25), we thus obtain

$$\mathrm{Id}_{\mu_{\mathsf{P}_1}(p_1)} = \mathrm{Id}_{\mu_{\mathsf{P}_2}(\Theta(p_1))} = \phi_{\mathsf{P}_1}(p_1, p_2)\,,$$

and so, finally,

$$p_2 = p_1.\phi_{\mathsf{P}_1}(p_1, p_2) = p_1.\mathrm{Id}_{\mu_{\mathsf{P}_1}(p_1)} = p_1\,,$$

which proves the injectivity of Θ.

Consider, next, an arbitrary point $p_2 \in \mathsf{P}_2$ over $\pi_{\mathsf{P}_2}(p_2) =: x \in \mathscr{M}$. Choose $q_1 \in \pi_{\mathsf{P}_1}^{-1}(\{x\})$. Clearly, $\pi_{\mathsf{P}_2}(\Theta(q_1)) = \pi_{\mathsf{P}_2}(p_2)$, and so

$$p_2 = \Theta(q_1).\phi_{\mathsf{P}_2}(\Theta(q_1), p_2)\,.$$

Upon invoking point (ii) of Proposition 8.28 and, once again, the property of Θ encoded in diagram (8.25), we establish

$$t(\phi_{\mathsf{P}_2}(\Theta(q_1), p_2)) = \mu_{\mathsf{P}_2}(\Theta(q_1)) = \mu_{\mathsf{P}_1}(q_1)\,.$$

Define

$$p_1 := q_1.\phi_{\mathsf{P}_2}(\Theta(q_1), p_2) \in \pi_{\mathsf{P}_1}^{-1}(\{x\})\,.$$

We then find, using the Gr-equivariance of Θ and property (i) of Proposition 8.28,

$$\Theta(p_1) = \Theta(q_1).\phi_{\mathsf{P}_2}(\Theta(q_1), p_2) = p_2\,,$$

which demonstrates the surjectivity of Θ and thus concludes the proof. □

A particularly powerful tool in the analysis of principal Gr-bundles, instrumental also in our subsequent discussion, is the local description, which we now set up after Moerdijk and Mrčun. The first prerequisite is described in

Definition 8.30. [MM03, Rem. 5.34(2)] Adopt the notation of Definitions 8.5 and 8.27. Let $\mathscr{M}, \mathscr{N}$ be a pair of smooth manifolds, $f : \mathscr{M} \to \mathscr{N}$ a smooth map between them, and $\mathcal{P}$ a principal Gr-bundle over $\mathscr{N}$. The **pullback** of $\mathcal{P}$ along f is the principal Gr-bundle over $\mathscr{M}$ given by

$$f^*\mathcal{P} := (f^*\mathsf{P}, \mathscr{M}, \mathrm{pr}_1, \mu_{\mathsf{P}} \circ \mathrm{pr}_2, \rho_{f^*\mathsf{P}})\,,$$

where

$$f^*\mathsf{P} := \mathscr{M}\,{}_f\!\times_{\pi_{\mathsf{P}}} \mathsf{P}\,,$$

and

$$\rho_{f^*\mathsf{P}} \;:\; f^*\mathsf{P}\,{}_{\mu_{\mathsf{P}} \circ \mathrm{pr}_2}\!\times_t \mathrm{Mor}\,\mathrm{Gr} \to f^*\mathsf{P} \;:\; ((m,p), \overrightarrow{g}) \mapsto (m, p.\overrightarrow{g})\,.$$

Clearly, the above definition makes sense, that is $f^*\mathcal{P}$ is a principal Gr-bundle. Indeed, properties (i) and (ii) from Definition 8.27 are manifest. As for the last property, we find, for any two points $(m_1, p_1), (m_2, p_2) \in f^*\mathsf{P}$ from the same fibre,

$$m_2 = \mathrm{pr}_1(m_2, p_2) = \mathrm{pr}_1(m_1, p_1) = m_1 \quad\Rightarrow\quad \pi_{\mathsf{P}}(p_2) = f(m_2) = f(m_1) = \pi_{\mathsf{P}}(p_1)\,,$$

and so, by virtue of Proposition 8.28,

$$p_2 = p_1.\phi_{\mathsf{P}}(p_1, p_2)\,.$$

Hence, the smooth inverse of the map

$$(\mathrm{pr}_1, \rho_{f^*\mathsf{P}}) \;:\; f^*\mathsf{P}\,{}_{\mu_{\mathsf{P}} \circ \mathrm{pr}_2}\!\times_t \mathrm{Mor}\,\mathrm{Gr} \to f^*\mathsf{P}\,{}_{\mathrm{pr}_1}\!\times_{\mathrm{pr}_1} f^*\mathsf{P} \equiv f^*\mathsf{P}^{[2]} \;:\; ((m,p), \overrightarrow{g}) \mapsto ((m,p), (m, p.\overrightarrow{g}))$$

reads

$$(\mathrm{pr}_1, \phi_{f^*\mathsf{P}}) \;:\; f^*\mathsf{P}^{[2]} \to f^*\mathsf{P}\,{}_{\mu_{\mathsf{P}} \circ \mathrm{pr}_2}\!\times_t \mathrm{Mor}\,\mathrm{Gr} \;:\; ((m,p_1), (m,p_2)) \mapsto ((m,p_1), \phi_{\mathsf{P}}(p_1,p_2))\,.$$

In the next step, we consider

Definition 8.31. [MM03, Rem. 5.34(1)] In the notation of Definition 8.5, the **unit bundle of** Gr is the principal Gr-bundle over $\mathrm{Ob}\,\mathrm{Gr}$ given by $\mathcal{U}_{\mathrm{Gr}} := (\mathrm{Mor}\,\mathrm{Gr}, \mathrm{Ob}\,\mathrm{Gr}, t, s, R)$, where R denotes right multiplication,

$$R \;:\; \mathrm{Mor}\,\mathrm{Gr}\,{}_s\!\times_t \mathrm{Mor}\,\mathrm{Gr} \to \mathrm{Mor}\,\mathrm{Gr} \;:\; (\overrightarrow{g}, \overrightarrow{h}) \mapsto \overrightarrow{g} \circ \overrightarrow{h}\,.$$

Once more, properties (i) and (ii) from Definition 8.27 are evident, and it remains to verify property (iii). The map

$$(\mathrm{pr}_1, R) \ : \ \mathrm{Mor}\,\mathrm{Gr}_s{\times}_t \mathrm{Mor}\,\mathrm{Gr} \to \mathrm{Mor}\,\mathrm{Gr}_t{\times}_t \mathrm{Mor}\,\mathrm{Gr} \ : \ (\overrightarrow{g}, \overrightarrow{h}) \mapsto (\overrightarrow{g}, \overrightarrow{g} \circ \overrightarrow{h})$$

admits the smooth inverse

$$(\mathrm{pr}_1, \phi_{\mathcal{U}_{\mathrm{Gr}}}) \ : \ \mathrm{Mor}\,\mathrm{Gr}_t{\times}_t \mathrm{Mor}\,\mathrm{Gr} \to \mathrm{Mor}\,\mathrm{Gr}_s{\times}_t \mathrm{Mor}\,\mathrm{Gr} \ : \ (\overrightarrow{g}, \overrightarrow{h}) \mapsto \big(\overrightarrow{g}, \overrightarrow{g}^{-1} \circ \overrightarrow{h}\big).$$

The last ingredient is

Definition 8.32. [MM03, Rem. 5.34(3)] In the notation of Definitions 8.5, 8.30 and 8.31, and for $\mathscr{M}$ a smooth manifold, a **trivial principal** Gr**-bundle** over $\mathscr{M}$ is the pullback $f^*\mathcal{U}_{\mathrm{Gr}}$ of the trivial bundle $\mathcal{U}_{\mathrm{Gr}}$ along an arbitrary smooth map $f : \mathscr{M} \to \mathrm{Ob}\,\mathrm{Gr}$.

We are now ready to state the important

Proposition 8.33. [MM03, Rem. 5.34(4)] *In the notation of Definitions 8.5 and 8.27, every principal* Gr*-bundle $\mathcal{P}$ is locally trivialisable, i.e. for every point $m \in \mathscr{M}$ of the base of $\mathcal{P}$, there exists an open neighbourhood $\mathcal{O} \ni m$ and a smooth map $\mu_{\mathcal{O}} : \mathcal{O} \to \mathrm{Ob}\,\mathrm{Gr}$ such that $\mathcal{P}|_{\mathcal{O}}$ is isomorphic to the trivial* Gr*-bundle $\mu_{\mathcal{O}}^*\mathcal{U}_{\mathrm{Gr}}$.*

Proof. Given a neighbourhood $\mathcal{O}$ of a point $m \in \mathscr{M}$, choose a smooth local section $\sigma_{\mathcal{O}} : \mathcal{O} \to \pi_{\mathsf{P}}^{-1}(\mathcal{O})$ of the surjective submersion $\pi_{\mathsf{P}} : \mathsf{P} \to \mathscr{M}$, and define the smooth map

$$\mu_{\mathcal{O}} := \mu_{\mathsf{P}} \circ \sigma_{\mathcal{O}}.$$

The map

$$\tau_{\mathcal{O}}^{-1} := \rho_{\mathsf{P}} \circ (\sigma_{\mathcal{O}} \times \mathrm{Id}_{\mathrm{Mor}\,\mathrm{Gr}}) \ : \ \mu_{\mathcal{O}}^*\mathrm{Mor}\,\mathrm{Gr} \to \pi_{\mathsf{P}}^{-1}(\mathcal{O}) \ : \ (m, \overrightarrow{g}) \mapsto \sigma_{\mathcal{O}}(m).\overrightarrow{g}$$

is manifestly well-defined as

$$(m, \overrightarrow{g}) \in \mu_{\mathcal{O}}^*\mathrm{Mor}\,\mathrm{Gr} \quad \Rightarrow \quad t(\overrightarrow{g}) = \mu_{\mathcal{O}}(m) \equiv \mu_{\mathsf{P}}\big(\sigma_{\mathcal{O}}(m)\big),$$

and smooth (as a composition of smooth maps). Viewed as a map between the two bundles, it preserves the respective fibres as

$$\pi_{\mathsf{P}} \circ \tau_{\mathcal{O}}^{-1}(m, \overrightarrow{g}) = \pi_{\mathsf{P}}\big(\sigma_{\mathcal{O}}(m).\overrightarrow{g}\big) = \pi_{\mathsf{P}} \circ \sigma_{\mathcal{O}}(m) = \mathrm{id}_{\mathcal{O}}(m) = m = \mathrm{pr}_1(m, \overrightarrow{g})$$

due to the Gr-equivariance of π_{P}. Moreover, it is itself Gr-equivariant, *i.e.* it renders the corresponding diagrams of Definition 8.26 commutative. Indeed, it preserves the momenta of the two bundles,

$$\mu_{\mathsf{P}} \circ \tau_{\mathcal{O}}^{-1}(m, \overrightarrow{g}) = \mu_{\mathsf{P}}\big(\sigma_{\mathcal{O}}(m).\overrightarrow{g}\big) = s(\overrightarrow{g}) = s \circ \mathrm{pr}_2(m, \overrightarrow{g})$$

(owing to the defining property (i) of μ_{P}), and it intertwines the respective right Gr-actions,

$$\begin{aligned}\rho_{\mathsf{P}} \circ (\tau_{\mathcal{O}}^{-1} \times \mathrm{Id}_{\mathrm{Mor}\,\mathrm{Gr}})\big((m, \overrightarrow{g}), \overrightarrow{h}\big) \quad &= \quad \rho_{\mathsf{P}}\big(\sigma_{\mathcal{O}}(m).\overrightarrow{g}, \overrightarrow{h}\big) = \big(\sigma_{\mathcal{O}}(m).\overrightarrow{g}\big).\overrightarrow{h} = \sigma_{\mathcal{O}}(m).(\overrightarrow{g} \circ \overrightarrow{h}) \\ &= \quad \tau_{\mathcal{O}}^{-1}(m, \overrightarrow{g} \circ \overrightarrow{h}) = \tau_{\mathcal{O}}^{-1}\big(m, R(\overrightarrow{g}, \overrightarrow{h})\big) = \tau_{\mathcal{O}}^{-1} \circ \rho_{\mu_{\mathcal{O}}^*\mathcal{U}_{\mathrm{Gr}}}\big((m, \overrightarrow{g}), \overrightarrow{h}\big).\end{aligned}$$

Thus, altogether, $\tau_{\mathcal{O}}^{-1}$ is a morphism between the two principal Gr-bundles over $\mathcal{O}$, and so – by virtue of Proposition 8.29 – it is an isomorphism. It is the inverse of the map

$$\tau_{\mathcal{O}}(p) = \big(\pi_{\mathsf{P}}(p), \phi_{\mathsf{P}}\big(\sigma_{\mathcal{O}} \circ \pi_{\mathsf{P}}(p), p\big)\big).$$

□

The above proposition paves the way to a local description of principal Gr-bundles that we shall find of great use in the context in hand. An abstraction of the hitherto findings yields

Definition 8.34. [Moe91, Sec. 1.2] Adopt the notation of Definitions 8.5 and 8.27. Let $\mathcal{O}_{\mathscr{M}} = \{\mathcal{O}_i\}_{i \in \mathscr{I}}$ be an open cover of $\mathscr{M}$ (with an index set $\mathscr{I}$), and let $\sigma_i : \mathcal{O}_i \to \pi_{\mathsf{P}}^{-1}(\mathcal{O}_i)$ be the associated local sections of the principal Gr-bundle $\mathcal{P}$ over $\mathscr{M}$. **Local (trivialising) data of $\mathcal{P}$ (with values in** Gr**)** are given by the triple $(\mathcal{O}_{\mathscr{M}}, \mu_i, \gamma_{ij} \mid i, j \in \mathscr{I})$ composed of two collections of smooth maps: **local momenta**

$$\mu_i := \mu_{\mathsf{P}} \circ \sigma_i \ : \ \mathcal{O}_i \to \mathrm{Ob}\,\mathrm{Gr},$$

and **transition maps**

$$\gamma_{ij} \ : \ \mathcal{O}_{ij} \to \mathrm{Mor}\,\mathrm{Gr} \ : \ m \mapsto \phi_{\mathsf{P}}\left(\sigma_i(m), \sigma_j(m)\right),$$

the latter being defined on non-empty double intersections $\mathcal{O}_{ij} = \mathcal{O}_i \cap \mathcal{O}_j$ and relating the respective restrictions of **local trivialisations**

$$\tau_i := (\pi_{\mathsf{P}}, \phi_{\mathsf{P}} \circ (\sigma_i \circ \pi_{\mathsf{P}}, \mathrm{id}_{\mathsf{P}})) \ : \ \pi_{\mathsf{P}}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mu_i^* \mathrm{Mor}\,\mathrm{Gr}\,.$$

Important properties of local trivialising data are listed in the following proposition that, at the same time, provides us with a key to understanding the geometry behind the data.

Proposition 8.35. [Moe91, Sec. 1.2][Ros04a, Lem. 3.7] *Adopt the notation of Definitions 8.5, 8.30, 8.31 and 8.34. Local trivialising data have the defining properties:*

(i) $t \circ \gamma_{ij} = \mu_i, \ s \circ \gamma_{ij} = \mu_j, \ \gamma_{ii} = \mathrm{Id} \circ \mu_i$;

(ii) $\gamma_{ji} = \mathrm{Inv} \circ \gamma_{ij}$;

(iii) *on a non-empty common intersection* $\mathcal{O}_{ijk} = \mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_k \ni m$ *of any three open sets* $\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k, \ i,j,k \in \mathscr{I}$*, the* ***cocycle condition*** $\gamma_{ik}(m) = \gamma_{ij}(m) \circ \gamma_{jk}(m)$ *obtains.*

They canonically define an isomorphism between the trivial bundles $\mu_i^* \mathcal{U}_{\mathrm{Gr}}|_{\mathcal{O}_{ij}}$ *and* $\mu_j^* \mathcal{U}_{\mathrm{Gr}}|_{\mathcal{O}_{ij}}$ *over every non-empty double intersection* $\mathcal{O}_{ij} = \mathcal{O}_i \cap \mathcal{O}_j$.

Proof. Properties (i), (ii) and (iii) are straightforward consequences of Proposition 8.28. Thus, it remains to prove the concluding statement. Consider the map

$$\varphi_{ij} \ : \ \mu_j^* \mathcal{U}_{\mathrm{Gr}}|_{\mathcal{O}_{ij}} \to \mu_i^* \mathcal{U}_{\mathrm{Gr}}|_{\mathcal{O}_{ij}} \ : \ (m, \overrightarrow{g}) \mapsto \big(m, \gamma_{ij}(m) \circ \overrightarrow{g}\big)\,.$$

The map is clearly well-defined as

$$(m, \overrightarrow{g}) \in \mu_j^* \mathcal{U}_{\mathrm{Gr}}|_{\mathcal{O}_{ij}} \quad \Rightarrow \quad t(\overrightarrow{g}) = \mu_j(m) = s \circ \gamma_{ij}(m)$$

owing to property (i) of the local data, and

$$t\big(\gamma_{ij}(m) \circ \overrightarrow{g}\big) = t\big(\gamma_{ij}(m)\big) = \mu_i(m) \quad \Rightarrow \quad \varphi_{ij}(m,g) \in \mu_i^* \mathcal{U}_{\mathrm{Gr}}\,.$$

Its surjectivity follows from the simple identity

$$\begin{aligned}(m, \overrightarrow{g}) \in \mu_i^* \mathcal{U}_{\mathrm{Gr}}|_{\mathcal{O}_{ij}} \quad \Rightarrow \quad (m, \overrightarrow{g}) \ &= \ (m, \mathrm{Id}_{t(\overrightarrow{g})} \circ \overrightarrow{g}) = (m, \mathrm{Id}_{\mu_i(m)} \circ \overrightarrow{g}) = \big(m, \gamma_{ii}(m) \circ \overrightarrow{g}\big) \\ &= \ \big(m, \gamma_{ij}(m) \circ \gamma_{ji}(m) \circ \overrightarrow{g}\big) = \varphi_{ij}\big(m, \gamma_{ji}(m) \circ \overrightarrow{g}\big)\,.\end{aligned}$$

The map is also manifestly fibre-preserving, and so it remains to check that it also preserves the momenta, which follows from

$$\mu_{\mathcal{O}_{ij}\,\mu_i\!\times_t \mathrm{Mor}\,\mathrm{Gr}} \circ \varphi_{ij}(m, \overrightarrow{g}) = s \circ \mathrm{pr}_2\big(m, \gamma_{ij}(m) \circ \overrightarrow{g}\big) = s(\overrightarrow{g}) = s \circ \mathrm{pr}_2(m, \overrightarrow{g}) = \mu_{\mathcal{O}_{ij}\,\mu_j\!\times_t \mathrm{Mor}\,\mathrm{Gr}}(m, \overrightarrow{g})\,,$$

and that it intertwines the two Gr-actions,

$$\varphi_{ij}\big((m, \overrightarrow{g}).\overrightarrow{h}\big) = \varphi_{ij}(m, \overrightarrow{g} \circ \overrightarrow{h}) = \big(m, \gamma_{ij}(m) \circ \overrightarrow{g} \circ \overrightarrow{h}\big) = \big(m, \gamma_{ij}(m) \circ \overrightarrow{g}\big).\overrightarrow{h} = \varphi_{ij}(m, \overrightarrow{g}).\overrightarrow{h}\,.$$

The claim is now implied by Proposition 8.29. □

The dependence of a local trivialisation on the choice of the local section is clarified by the following

Proposition 8.36. [Ros04a, Lem. 3.2] *Adopt the notation of Definitions 8.5, 8.30, 8.31 and 8.34. Let* $\sigma^i_{\mathcal{O}} : \mathcal{O} \to \mathsf{P}, \ i \in \{1,2\}$ *be any two smooth local sections of a principal* Gr*-bundle* $\mathcal{P}$ *over an open subset* $\mathcal{O} \subset \mathscr{M}$ *of the base* $\mathscr{M}$ *of* $\mathcal{P}$*. The associated local trivialisations* $\tau^i_{\mathcal{O}}$ *of* $\mathcal{P}$ *over* $\mathcal{O}$ *are equivalent in the sense that the corresponding trivial bundles* $\mu^{i\,*}_{\mathcal{O}} \mathcal{U}_{\mathrm{Gr}}$*, defined in terms of the respective local momenta* $\mu^i_{\mathcal{O}}$ *associated with the* $\sigma^i_{\mathcal{O}}$*, are isomorphic as per*

$$\tau^{2,1}_{\mathcal{O}} := \tau^2_{\mathcal{O}} \circ \big(\tau^1_{\mathcal{O}}\big)^{-1} \ : \ \mu^{1\,*}_{\mathcal{O}} \mathcal{U}_{\mathrm{Gr}} \xrightarrow{\cong} \mu^{2\,*}_{\mathcal{O}} \mathcal{U}_{\mathrm{Gr}}\,.$$

Proof. The claim follows from Proposition 8.29 and the proven Gr-equivariance of the local trivialisations. □

We conclude our introductory presentation of principal Lie-groupoid bundles by formulating a variant of the familiar clutching construction.

Theorem 8.37. [Moe91, Sec. 1.2][Ros04a, Thm. 3.8] *Adopt the notation of Definitions 8.5, 8.30, 8.31 and 8.34. Define a manifold*

$$\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}} \coloneqq \bigsqcup_{i\in\mathscr{I}} \mu_i^*\mathcal{U}_{\rm Gr}/\sim_{(\gamma_{ij})}$$

as the quotient with respect to the equivalence relation

$$(i,m_i,\vec{g}_i)\sim_{(\gamma_{ij})}(j,m_j,\vec{g}_j) \quad\Leftrightarrow\quad \big(\ m_i=m_j\in\mathcal{O}_{ij} \quad\wedge\quad \vec{g}_i=\gamma_{ij}(m_i)\circ\vec{g}_j\ \big)\,, \tag{8.28}$$

and maps

$$\begin{aligned}
\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}} &: \quad \vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}\to{\rm Ob\,Gr} \ :\ \big[(i,m,\vec{g})\big]\mapsto s(\vec{g}) \\
\pi_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}} &: \quad \vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}\to\mathscr{M} \ :\ \big[(i,m,\vec{g})\big]\mapsto m\,, \\
\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}} &: \quad \vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}\,{}_{\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}}\!\times_t{\rm Mor\,Gr}\to\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}} \ :\ \big(\big[(i,m,\vec{g})\big],\vec{h}\big)\mapsto\big[(i,m,\vec{g}\circ\vec{h})\big]\,.
\end{aligned}$$

The quintuple

$$\vec{\mathcal{P}}_{\mathcal{O}_{\mathscr{M}}}\coloneqq\big(\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}},\mathscr{M},\pi_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}},\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}},\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}\big)$$

is a principal Gr*-bundle over* $\mathscr{M}$*.*

Proof. First of all, we convince ourselves that relation (8.28) is an equivalence relation using the defining properties of the γ_{ij}. The space $\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}$ is then a smooth quotient of smooth spaces, locally diffeomorphic to $\mu_i^*\mathcal{U}_{\rm Gr}$, and the map $\pi_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}$ is a surjective submersion. What has to be shown is that the smooth map $\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}$ endows the triple $(\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}},\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}},\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})$ with the structure of a right Gr-module, that $\pi_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}$ is invariant under the Gr-action, and that the map $({\rm pr}_1,\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})$ is a diffeomorphism so that the Gr-action is free and transitive.

The former fact follows straightforwardly from the simple identities

$$\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}\big(\big[(i,m,\vec{g})\big].\vec{h}\big)=\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}\big(\big[(i,m,\vec{g}\circ\vec{h})\big]\big)=s(\vec{g}\circ\vec{h})=s(\vec{h})\,,$$

$$\big[(i,m,\vec{g})\big].{\rm Id}_{\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}([(i,m,\vec{g})])}=\big[(i,m,\vec{g}\circ{\rm Id}_{\mu_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}([(i,m,\vec{g})])})\big]=\big[(i,m,\vec{g}\circ{\rm Id}_{s(\vec{g})})\big]=\big[(i,m,\vec{g})\big]\,,$$

$$\big(\big[(i,m,\vec{g})\big].\vec{h}_1\big).\vec{h}_2=\big[(i,m,\vec{g}\circ\vec{h}_1)\big].\vec{h}_2=\big[(i,m,\vec{g}\circ\vec{h}_1\circ\vec{h}_2)\big]=\big[(i,m,\vec{g})\big].(\vec{h}_1\circ\vec{h}_2)\,,$$

and the Gr-invariance of $\pi_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}}$ is self-evident.

In order to prove the latter fact, note that a pair $\big(\big[(i,m_i,\vec{g}_i)\big],\big[(j,m_j,\vec{g}_j)\big]\big)$ with a common projection to $\mathscr{M}$, that is with $m_j=m_i\eqqcolon m\in\mathcal{O}_{ij}$, unambiguously defines a morphism

$$\vec{h}(m)\coloneqq\vec{g}_i^{-1}\circ\gamma_{ij}(m)\circ\vec{g}_j\,.$$

The definition makes sense as

$$t\big(\gamma_{ij}(m)\big)=\mu_i(m)=t(\vec{g}_i)=s(\vec{g}_i^{-1})\,,\qquad\qquad s\big(\gamma_{ij}(m)\big)=\mu_j(m)=t(\vec{g}_j)\,,$$

and it is independent of the choice of representatives of the two equivalence classes. Indeed, for $\big[(k,m_k,\vec{g}_k)\big]=\big[(i,m_i,\vec{g}_i)\big]$ and $\big[(l,m_l,\vec{g}_l)\big]=\big[(j,m_j,\vec{g}_j)\big]$ (with, necessarily, $m_k=m_l=m\in\mathcal{O}_{ijkl}$), we find

$$\begin{aligned}
\vec{g}_k^{-1}\circ\gamma_{kl}(m)\circ\vec{g}_l &= \big(\gamma_{ki}(m)\circ\vec{g}_i\big)^{-1}\circ\gamma_{kl}(m)\circ\big(\gamma_{lj}(m)\circ\vec{g}_j\big)=\vec{g}_i^{-1}\circ\gamma_{ik}(m)\circ\gamma_{kl}(m)\circ\gamma_{lj}(m)\circ\vec{g}_j \\
&= \vec{g}_i^{-1}\circ\gamma_{ij}(m)\circ\vec{g}_j\,.
\end{aligned}$$

We may then write down the smooth inverse of $({\rm pr}_1,\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})$ in the form

$$({\rm pr}_1,\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})^{-1}\big(\big[(i,m_i,\vec{g}_i)\big],\big[(j,m_j,\vec{g}_j)\big]\big)\coloneqq\big(\big[(i,m_i,\vec{g}_i)\big],\vec{h}(m_i)\big)$$

and check its desired properties:

$$\begin{aligned}
&({\rm pr}_1,\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})\circ({\rm pr}_1,\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})^{-1}\big(\big[(i,m_i,\vec{g}_i)\big],\big[(j,m_j,\vec{g}_j)\big]\big)=({\rm pr}_1,\rho_{\vec{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})\big(\big[(i,m_i,\vec{g}_i)\big],\vec{h}(m_i)\big) \\
&\quad=\big(\big[(i,m_i,\vec{g}_i)\big],\big[(i,m_i,\vec{g}_i\circ\vec{g}_i^{-1}\circ\gamma_{ij}(m)\circ\vec{g}_j)\big]\big)=\big(\big[(i,m_i,\vec{g}_i)\big],\big[(i,m_i,\gamma_{ij}(m)\circ\vec{g}_j)\big]\big) \\
&\quad=\big(\big[(i,m_i,\vec{g}_i)\big],\big[(i,m_j,\vec{g}_j)\big]\big)
\end{aligned}$$

and

$$(\mathrm{pr}_1,\rho_{\bar{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})^{-1}\circ(\mathrm{pr}_1,\rho_{\bar{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})\big(\big[(i,m_i,\vec{g}_i)\big],\vec{g}\big)=(\mathrm{pr}_1,\rho_{\bar{\mathsf{P}}_{\mathcal{O}_{\mathscr{M}}}})^{-1}\big(\big[(i,m_i,\vec{g}_i)\big],\big[(i,m_i,\vec{g}_i\circ\vec{g})\big]\big)$$
$$=\big(\big[(i,m_i,\vec{g}_i)\big],\vec{g}_i^{-1}\circ\gamma_{ii}(m_i)\circ(\vec{g}_i\circ\vec{g})\big)=\big(\big[(i,m_i,\vec{g}_i)\big],\vec{g}\big)\,.$$

□

The local language can equally well be developed for morphisms between principal Lie-groupoid bundles. We begin with

Proposition 8.38. *Adopt the notation of Definitions 8.5 and 8.34. Let $\mathcal{P}_A, A\in\{1,2\}$ be a pair of principal* Gr*-bundles with the respective local bundle data $(\mathcal{O}_{\mathscr{M}},\mu_i^A,\gamma_{ij}^A\,|\,i,j\in\mathscr{I})$. The existence of a morphism $\Theta\in\mathrm{Hom}_{\mathrm{Gr}\text{-}\mathfrak{Bun}(\mathscr{M})}(\mathcal{P}_1,\mathcal{P}_2)$ is equivalent to the existence of a collection of locally smooth maps $\theta_i:\mathcal{O}_i\to\mathrm{Mor}\,\mathrm{Gr}$ with the defining properties:*

(i) $s\circ\theta_i=\mu_i^1,\ t\circ\theta_i=\mu_i^2$;
(ii) *on a non-empty common intersection $\mathcal{O}_{ij}=\mathcal{O}_i\cap\mathcal{O}_j\ni m$ of any two open sets $\mathcal{O}_i,\mathcal{O}_j,\ i,j\in\mathscr{I}$, the* ***intertwiner condition*** *$\theta_i(m)\circ\gamma_{ij}^1(m)=\gamma_{ij}^2(m)\circ\theta_j(m)$ obtains.*

Proof. First, consider local trivialisations $\tau_i^A:\pi_{\mathsf{P}_A}^{-1}(\mathcal{O}_i)\to\mu_i^{A\,*}\mathrm{Mor}\,\mathrm{Gr}$ associated with the local data $(\mathcal{O}_{\mathscr{M}},\mu_i^A,\gamma_{ij}^A\mid i,j\in\mathscr{I})$, and the corresponding local sections $\sigma_i^A:\mathcal{O}_i\to\mathsf{P}_A$. Denote by ϕ_{P_A} the respective division maps. As Θ preserves fibres, we necessarily find, for any $m\in\mathcal{O}_i$,

$$\tau_i^2\circ\Theta\circ\sigma_i^1(m)=\big(m,\phi_{\mathsf{P}_2}\big(\sigma_i^2(m),\Theta\circ\sigma_i^1(m)\big)\big)\,,$$

and so

$$\Theta\circ\sigma_i^1(m)=\sigma_i^2(m).\phi_{\mathsf{P}_2}\big(\sigma_i^2(m),\Theta\circ\sigma_i^1(m)\big)\,.$$

Define

$$\theta_i\ :\ \mathcal{O}_i\to\mathrm{Mor}\,\mathrm{Gr}\ :\ m\mapsto\phi_{\mathsf{P}_2}\big(\sigma_i^2(m),\Theta\circ\sigma_i^1(m)\big)\,.$$

Adducing point (ii) of Proposition 8.28 and subsequently using the defining property of a Gr-bundle morphism encoded in diagram (8.25), we readily find the desired identity

$$s\circ\theta_i(m)=\mu_{\mathsf{P}_2}\circ\Theta\circ\sigma_i^1(m)=\mu_{\mathsf{P}_1}\circ\sigma_i^1(m)\equiv\mu_i^1(m)\,.$$

Next, once more with the help of point (ii) of Proposition 8.28, we obtain

$$t\circ\theta_i(m)=\mu_{\mathsf{P}_2}\circ\sigma_i^2(m)\equiv\mu_i^2(m)\,.$$

Finally, the intertwiner property of Θ captured by diagram (8.26), enables us to demonstrate the validity of point (ii),

$$\begin{aligned}\sigma_j^2(m).\big(\theta_j(m)\circ\gamma_{ji}^1(m)\big)&\equiv\big(\tau_j^2\big)^{-1}(m,\mathrm{Id}_{\mu_j^2(m)}).\big(\theta_j(m)\circ\gamma_{ji}^1(m)\big)=\Big(\big(\tau_j^2\big)^{-1}(m,\mathrm{Id}_{\mu_j^2(m)}).\theta_j(m)\Big).\gamma_{ji}^1(m)\\&=\Theta\circ\big(\tau_j^1\big)^{-1}(m,\mathrm{Id}_{\mu_j^1(m)}).\gamma_{ji}^1(m)=\Theta\circ\big(\tau_j^1\big)^{-1}\big(m,\gamma_{ji}^1(m)\big)\\&=\Theta\circ\big(\tau_j^1\big)^{-1}\circ\tau_j^1\circ\big(\tau_i^1\big)^{-1}(m,\mathrm{Id}_{\mu_i^1(m)})\equiv\Theta\circ\big(\tau_i^1\big)^{-1}(m,\mathrm{Id}_{\mu_i^1(m)})\\&=\big(\tau_i^2\big)^{-1}(m,\theta_i(m))=\big(\tau_j^2\big)^{-1}\circ\tau_j^2\circ\big(\tau_i^2\big)^{-1}(m,\mathrm{Id}_{\mu_i^2(m)}).\theta_i(m)\\&=\sigma_j^2(m).\big(\gamma_{ji}^2(m)\circ\theta_j(m)\big)\,.\end{aligned}$$

Here, we are using the fact that the defining Gr-action on P_2 is free.

Conversely, let (θ_i) be a collection of locally smooth maps satisfying conditions (i) and (ii). We shall demonstrate that they induce Gr-bundle (iso)morphisms $\widetilde{\theta}_i:\mu_i^{1\,*}\mathcal{U}_{\mathrm{Gr}}\xrightarrow{\cong}\mu_i^{2\,*}\mathcal{U}_{\mathrm{Gr}}$ between the local trivialisations of the principal Gr-bundles $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Define a smooth map

$$\widetilde{\theta}_i\ :\ \mathscr{M}\,{}_{\mu_i^1}\!\times_t\mathrm{Mor}\,\mathrm{Gr}\to\mathscr{M}\,{}_{\mu_i^2}\!\times_t\mathrm{Mor}\,\mathrm{Gr}\ :\ (m,\vec{g})\mapsto(m,\theta_i(m)\circ\vec{g})\,.$$

The definition makes sense as for $(m,\vec{g})$ such that $\mu_i^1(m)=t(\vec{g})$ we have

$$s\left(\theta_i(m)\right)=\mu_i^1(m)=t(\vec{g})$$

and

$$t\left(\theta_i(m)\circ\vec{g}\right)=t\circ\theta_i(m)=\mu_i^2(m)\,.$$

The map is surjective,

$$(m,\vec{g})\in\mathscr{M}_{\mu_i^2}\times_t\mathrm{Mor\,Gr}\quad\Rightarrow\quad(m,\vec{g})=\widetilde{\theta}_i\big(m,\theta_i(m)^{-1}\circ\vec{g}\big)\,,$$

preserves fibres,

$$\pi_{\mu_i^2{}^*\mathrm{Mor\,Gr}}\circ\widetilde{\theta}_i(m,\vec{g})\equiv\mathrm{pr}_1\,(m,\theta_i(m)\circ\vec{g})=m=\mathrm{pr}_1(m,\vec{g})\equiv\pi_{\mu_i^1{}^*\mathrm{Mor\,Gr}}(m,\vec{g})\,,$$

intertwines the momenta,

$$\mu_{\mu_i^2{}^*\mathrm{Mor\,Gr}}\circ\widetilde{\theta}_i(m,\vec{g})\equiv s\circ\mathrm{pr}_2\,(m,\theta_i(m)\circ\vec{g})=s(\vec{g})=s\circ\mathrm{pr}_2(m,\vec{g})\equiv\mu_{\mu_i^1{}^*\mathrm{Mor\,Gr}}(m,\vec{g})\,,$$

and is manifestly (right-)Gr-equivariant,

$$\widetilde{\theta}_i\big((m,\vec{g}).\vec{h}\big)\equiv\widetilde{\theta}_i(m,\vec{g}\circ\vec{h})=\big(m,\theta_i(m)\circ\vec{g}\circ\vec{h}\big)=(m,\theta_i(m)\circ\vec{g})\,.\vec{h}\equiv\widetilde{\theta}_i(m,\vec{g}).\vec{h}\,,$$

which altogether means that it is a Gr-bundle morphism, and hence an isomorphism. □

Theorem 8.37 and Proposition 8.38 permit to reduce the analysis of the category Gr-$\mathfrak{Bun}(\mathscr{M})$ to that of local data for its objects and morphisms. We shall use this fact below in an explicit discussion of the case of interest, which is that of the action groupoid $\mathrm{Gr}=\mathrm{G}\ltimes M$ introduced earlier. By way of preparation, we formulate

Definition 8.39. Adopt the notation of Definitions 8.5 and 8.8. Let Σ be a smooth space, G a Lie group, and $\mathscr{M}$ a smooth G-space. Denote by G-$\mathfrak{Bun}(\Sigma)$ the groupoid of principal G-bundles with base Σ. The **groupoid G-$\mathfrak{Bun}(\Sigma\,\|\,\mathscr{M})$ of principal G-bundles with base Σ gauging $\mathscr{M}$** is the subgroupoid of G-$\mathfrak{Bun}(\Sigma)$ composed of the objects $\mathcal{P}_{\mathrm{G}}=(\mathsf{P}_{\mathrm{G}},\Sigma,\pi_{\mathsf{P}_{\mathrm{G}}},\rho_{\mathsf{P}_{\mathrm{G}}})$ of the latter category (and all morphisms between them) with the property that the corresponding associated bundles $\mathsf{P}_{\mathrm{G}}\times_{\mathrm{G}}\mathscr{M}\to\Sigma$ admit a global section.

Remark 8.40. The definition makes sense as every isomorphism $\chi\in\mathrm{Hom}_{\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma)}(\mathcal{P}^1_{\mathrm{G}},\mathcal{P}^2_{\mathrm{G}})$ between bundles $\mathcal{P}^1_{\mathrm{G}},\mathcal{P}^2_{\mathrm{G}}\in\mathrm{Ob}\,\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma\,\|\,\mathscr{M})$ canonically induces an isomorphism $\widetilde{\chi}\in\mathrm{Hom}_{\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma\,\|\,\mathscr{M})}(\mathcal{P}^1_{\mathrm{G}},\mathcal{P}^2_{\mathrm{G}})$. This is readily verified in the local description associated with a choice $\mathcal{O}_\Sigma:=\{\Sigma_i\}_{i\in\mathscr{I}}$ of an open cover of the common base Σ of the two bundles in which χ, being an invertible fibre-preserving G-map $\chi:\mathsf{P}^1_{\mathrm{G}}\to\mathsf{P}^2_{\mathrm{G}}$, is described by a collection of locally smooth maps $\chi_i:\Sigma_i\to\mathrm{G}$ defined by the formulæ

$$\tau_i^2\circ\chi\circ\sigma_i^1(\sigma)=:(\sigma,\chi_i(\sigma))\,,\tag{8.29}$$

written for $\sigma\in\Sigma_i$ and for a local section $\sigma_i^1(\sigma)=\big(\tau_i^1\big)^{-1}(\sigma,e)$ determined by a local trivialisation $\tau_i^1:\pi^{-1}_{\mathsf{P}^1_{\mathrm{G}}}(\Sigma_i)\to\Sigma_i\times\mathrm{G}$, and satisfying the identity

$$g^2_{ij}(\sigma)=\chi_i(\sigma)\cdot g^1_{ij}(\sigma)\cdot\chi_j(\sigma)^{-1}\tag{8.30}$$

written for $\sigma\in\Sigma_{ij}$ and the transition maps $g^A_{ij}:\Sigma_{ij}\to\mathrm{G}$ of $\mathsf{P}^A_{\mathrm{G}}$, the latter being defined by the relation

$$\tau_i^A\circ\big(\tau_j^A\big)^{-1}(\sigma,e)=:\big(\sigma,g^A_{ij}(\sigma)\big)\,.$$

This follows from a specialisation of Proposition 8.38 to the case of the Lie group G viewed as a groupoid with the object manifold given by a singleton $\{\bullet\}$.

Consider, now, a global section $\eta^1\in\Gamma(\mathsf{P}^1_{\mathrm{G}}\times_{\mathrm{G}}\mathscr{M})$ with

$$\eta^1\;:\;\Sigma_i\to\mathsf{P}^1_{\mathrm{G}}\times_{\mathrm{G}}\mathscr{M}\;:\;\sigma\mapsto\big[\big(\sigma_i^1(\sigma),m_i(\sigma)\big)\big]$$

such that, for any $\sigma\in\Sigma_{ij}$, we obtain

$$\big[\big(\sigma_j^1(\sigma),m_j(\sigma)\big)\big]=\big[\big(\sigma_i^1(\sigma),m_i(\sigma)\big)\big]\,.$$

The left-hand side equals

$$\big[\big(\sigma_j^1(\sigma),m_j(\sigma)\big)\big]=\big[\big(\sigma_i^1(\sigma).g^1_{ij}(\sigma),m_j(\sigma)\big)\big]=\big[\big(\sigma_i^1(\sigma),g^1_{ij}(\sigma).m_j(\sigma)\big)\big]\,,$$

and so we must require that the gluing condition

$$m_i(\sigma)=g^1_{ij}(\sigma).m_j(\sigma)$$

hold true over Σ_{ij}. The existence of the locally smooth maps $m_i:\Sigma_i\to\mathscr{M}$ is thus tantamount to the existence of a global section of the associated bundle $\mathsf{P}^1_{\mathrm{G}}\times_{\mathrm{G}}\mathscr{M}$. Define

$$\widetilde{\chi}(\eta^1)(\sigma):=\big[\big(\sigma_i^2(\sigma),\chi_i(\sigma).m_i(\sigma)\big)\big]\,,$$

where, as usual, the local section $\sigma_i^2(\sigma) = \left(\tau_i^2\right)^{-1}(\sigma, e)$ is defined in terms of the very same local trivialisation τ_i^2 as the one entering the definition of the χ_i. Clearly, $\left[\left(\sigma_i^2(\cdot), \chi_i(\cdot).m_i(\cdot)\right)\right]$ is a local section of $\mathsf{P}^2_{\mathrm{G}} \times_{\mathrm{G}} \mathscr{M}$ over Σ_i, from which it follows that $\widetilde{\chi}(\eta^1)$ is a collection of local sections of the associated bundle $\mathsf{P}^2_{\mathrm{G}} \times_{\mathrm{G}} \mathscr{M}$. We readily convince ourselves that this last section is global,

$$\begin{aligned}
\left[\left(\sigma_j^2(\sigma), \chi_j(\sigma).m_j(\sigma)\right)\right] &= \left[\left(\left(\tau_i^2\right)^{-1} \circ \tau_i^2 \circ \left(\tau_j^2\right)^{-1}(\sigma, e), \left(\chi_j(\sigma) \cdot g^1_{ji}(\sigma)\right).m_i(\sigma)\right)\right] \\
&= \left[\left(\sigma_i^2(\sigma).g^2_{ij}(\sigma), \left(\chi_j(\sigma) \cdot g^1_{ji}(\sigma)\right).m_i(\sigma)\right)\right] \\
&= \left[\left(\sigma_i^2(\sigma), \left(g^2_{ij}(\sigma) \cdot \chi_j(\sigma) \cdot g^1_{ji}(\sigma)\right).m_i(\sigma)\right)\right] \\
&= \left[\left(\sigma_i^2(\sigma), \chi_i(\sigma).m_i(\sigma)\right)\right] ,
\end{aligned}$$

as stipulated by the definition.

We come to the main result of our considerations.

Theorem 8.41. *Adopt the notation of Definitions 8.5, 8.27 and 8.39. There exists an isomorphism of groupoids*

$$\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma \,\|\, \mathscr{M}) \cong \mathrm{G}{\ltimes}\mathscr{M}\text{-}\mathfrak{Bun}(\Sigma) .$$

Proof. By way of a proof, we give an explicit construction of a essentially surjective fully faithful functor

$$Gr \;:\; \mathrm{G}\text{-}\mathfrak{Bun}(\Sigma \,\|\, \mathscr{M}) \to \mathrm{G}{\ltimes}\mathscr{M}\text{-}\mathfrak{Bun}(\Sigma)$$

in the local description of both (small) categories. Thus, as the point of departure of our construction we take an open cover $\{\Sigma_i\}_{i \in \mathscr{I}} =: \mathcal{O}_\Sigma$ of Σ, to which we associate local data of principal G-bundles, bundles associated to them, principal $\mathrm{G}{\ltimes}\mathscr{M}$-bundles, and (iso)morphisms between them.

Take a principal G-bundle $\mathcal{P}_{\mathrm{G}} = (\mathsf{P}_{\mathrm{G}}, \Sigma, \pi_{\mathsf{P}_{\mathrm{G}}}, \rho_{\mathsf{P}_{\mathrm{G}}})$ with local trivialisations

$$\tau_i \;:\; \pi^{-1}_{\mathsf{P}_{\mathrm{G}}}(\Sigma_i) \to \Sigma_i \times \mathrm{G}$$

and transition maps

$$\tau_{ij}(\sigma, e) = \tau_i \circ \tau_j^{-1}(\sigma, e) = (\sigma, g_{ij}(\sigma)) ,$$

written in terms of a Čech 1-cocycle $g_{ij} : \Sigma_{ij} \to \mathrm{G}$. Form the associated bundle $\mathsf{P}_{\mathrm{G}} \times_{\mathrm{G}} \mathscr{M} \to \Sigma$ and assume the existence of a global section

$$\eta \;:\; \Sigma \to \mathsf{P}_{\mathrm{G}} \times_{\mathrm{G}} \mathscr{M}$$

with restrictions

$$\eta \;:\; \Sigma_i \to \mathsf{P}_{\mathrm{G}} \times_{\mathrm{G}} \mathscr{M} \;:\; \sigma \mapsto \left[\left(\tau_i^{-1}(\sigma, e), m_i(\sigma)\right)\right],$$

written in terms of some locally smooth maps $m_i : \Sigma_i \to \mathscr{M}$ that satisfy the relation

$$m_i(\sigma) = g_{ij}(\sigma).m_j(\sigma)$$

over double intersections $\Sigma_{ij} \ni \sigma$.

To $\mathcal{P}_{\mathrm{G}}$, we associate a principal $\mathrm{G}{\ltimes}\mathscr{M}$-bundle as follows: Define locally smooth maps

$$\mu_i \;:\; \Sigma_i \to \mathscr{M} \;:\; \sigma \mapsto m_i(\sigma) , \qquad \gamma_{ij} \;:\; \Sigma_{ij} \to \mathrm{G} \times \mathscr{M} \;:\; \sigma \mapsto (g_{ij}(\sigma), m_j(\sigma)) .$$

These satisfy the identities

$$t \circ \gamma_{ij}(\sigma) = g_{ij}(\sigma).m_j(\sigma) = m_i(\sigma) \equiv \mu_i(\sigma) , \qquad s \circ \gamma_{ij}(\sigma) = m_j(\sigma) \equiv \mu_j(\sigma) ,$$

$$\gamma_{ii}(\sigma) = (e, m_i(\sigma)) \equiv \mathrm{Id}_{m_i(\sigma)} \equiv \mathrm{Id} \circ \mu_i(\sigma) ,$$

$$\gamma_{ji}(\sigma) = \left(g_{ij}(\sigma)^{-1}, m_i(\sigma)\right) = \left(g_{ij}(\sigma)^{-1}, g_{ij}(\sigma).m_j(\sigma)\right) \equiv \mathrm{Inv} \circ \gamma_{ij}(\sigma) ,$$

$$\gamma_{ij}(\sigma) \circ \gamma_{jk}(\sigma) = (g_{ij}(\sigma), g_{jk}(\sigma).m_k(\sigma)) \circ (g_{jk}(\sigma), m_k(\sigma)) = (g_{ij}(\sigma) \cdot g_{jk}(\sigma), m_k(\sigma)) = \gamma_{ik}(\sigma) ,$$

and so we conclude that the collection $\left(\mathcal{O}_\Sigma, m_i, (g_{ij}, m_j) \mid i, j \in \mathscr{I}\right)$ defines local data of a principal $\mathrm{G}{\ltimes}\mathscr{M}$-bundle over Σ. Upon applying the clutching construction of Theorem 8.37, we thus obtain the total space

$$\vec{\mathsf{P}}_{\mathcal{O}_\Sigma} := \bigsqcup_{i \in \mathscr{I}} m_i^* \mathcal{U}_{\mathrm{G}{\ltimes}\mathscr{M}} / \sim_{(g_{ij}, m_j)}$$

of a principal G⋉$\mathscr{M}$-bundle which we declare to be the Gr-image of $\mathcal{P}_{\rm G}$,

$$Gr(\mathcal{P}_{\rm G}) := \left(\vec{\mathsf{P}}_{\mathcal{O}_\Sigma}, \Sigma, \pi_{\vec{\mathsf{P}}_{\mathcal{O}_\Sigma}}, \mu_{\vec{\mathsf{P}}_{\mathcal{O}_\Sigma}}, \rho_{\vec{\mathsf{P}}_{\mathcal{O}_\Sigma}}\right).$$

Next, we shall verify that the above assignment is functorial by associating morphisms between G⋉$\mathscr{M}$-bundles to those between G-bundles gauging $\mathscr{M}$. To this end, consider a pair $\mathcal{P}^1_{\rm G}, \mathcal{P}^2_{\rm G} \in {\rm Ob}\,{\rm G}\text{-}\mathfrak{Bun}(\Sigma \,\|\, \mathscr{M})$ and assume given local data (χ_i) (associated with $\mathcal{O}_\Sigma$) of an isomorphism $\chi : \mathcal{P}^1_{\rm G} \xrightarrow{\cong} \mathcal{P}^2_{\rm G}$ determined by the relations

$$\tau^2_i \circ \chi \circ \left(\tau^1_i\right)^{-1}(\sigma, e) = (\sigma, \chi_i(\sigma)),$$

written, for $\sigma \in \Sigma_i$, in terms of local trivialisations $\tau^A_i : \pi^{-1}_{\mathsf{P}^A_{\rm G}}(\Sigma_i) \to \Sigma_i \times {\rm G}$, $A \in \{1,2\}$. The corresponding global sections of the associated bundles are related as described in Remark 8.40, that is

$$\eta^1 \;:\; \Sigma_i \to \pi^{-1}_{\mathsf{P}^1_{\rm G}\times_{\rm G}\mathscr{M}}(\Sigma_i) \;:\; \sigma \mapsto \left[\left(\left(\tau^1_i\right)^{-1}(\sigma,e), m_i(\sigma)\right)\right]$$

is mapped to

$$\widetilde{\chi}(\eta^1) \;:\; \Sigma_i \to \pi^{-1}_{\mathsf{P}^2_{\rm G}\times_{\rm G}\mathscr{M}}(\Sigma_i) \;:\; \sigma \mapsto \left[\left(\left(\tau^2_i\right)^{-1}(\sigma,e), \chi_i(\sigma).m_i(\sigma)\right)\right]$$

by the induced isomorphism $\widetilde{\chi}$. Accordingly, we find

$$\vec{\mathsf{P}}^1_{\mathcal{O}_\Sigma} = \bigsqcup_{i\in\mathscr{I}} m_i^*\mathcal{U}_{{\rm G}\ltimes\mathscr{M}} / \sim_{(g^1_{ij}, m_j)}$$

and

$$\vec{\mathsf{P}}^2_{\mathcal{O}_\Sigma} = \bigsqcup_{i\in\mathscr{I}} (\chi_i.m_i)^*\mathcal{U}_{{\rm G}\ltimes\mathscr{M}} / \sim_{(\chi_i\cdot g^1_{ij}\cdot({\rm Inv}\circ\chi_j), \chi_j.m_j)}$$

as total spaces of $Gr(\mathcal{P}^1_{\rm G})$ and $Gr(\mathcal{P}^2_{\rm G})$, respectively, and so we conclude that the desired isomorphism

$$Gr(\chi) \;:\; Gr(\mathcal{P}^1_{\rm G}) \xrightarrow{\cong} Gr(\mathcal{P}^2_{\rm G})$$

is determined by local data

$$\theta_i := (\chi_i, m_i) \equiv (\chi_i, \mu^1_i).$$

Indeed, we obtain

$$s \circ \theta_i(\sigma) = m_i(\sigma) \equiv \mu^1_i(\sigma), \qquad t \circ \theta_i(\sigma) = \chi_i(\sigma).m_i(\sigma) \equiv \mu^2_i(\sigma)$$

and

$$\begin{aligned}
\theta_i(\sigma) \circ \gamma^1_{ij}(\sigma) \circ \theta_j(\sigma)^{-1} &\equiv (\chi_i(\sigma), m_i(\sigma)) \circ \left(g^1_{ij}(\sigma), m_j(\sigma)\right) \circ \left(\chi_j(\sigma)^{-1}, \chi_j(\sigma).m_j(\sigma)\right) \\
&= \left(\chi_i(\sigma), g^1_{ij}(\sigma).m_j(\sigma)\right) \circ \left(g^1_{ij}(\sigma), m_j(\sigma)\right) \circ \left(\chi_j(\sigma)^{-1}, \chi_j(\sigma).m_j(\sigma)\right) \\
&= \left(\chi_i(\sigma) \circ g^1_{ij}(\sigma) \circ \chi_j(\sigma)^{-1}, \chi_j(\sigma).m_j(\sigma)\right) \\
&= \gamma^2_{ij}(\sigma),
\end{aligned}$$

in conformity with Proposition 8.38. It is clear from the very definition of the mapping Gr that its morphism component preserves composition of morphisms as for

$$\mathcal{P}^1_{\rm G} \xrightarrow{\chi} \mathcal{P}^2_{\rm G} \xrightarrow{\chi'} \mathcal{P}^3_{\rm G}$$

we get

$$Gr(\chi' \circ \chi) = (\chi'_i \cdot \chi_i, \mu^1_i) = (\chi'_i, \chi_i.\mu^1_i) \circ (\chi_i, \mu^1_i) \equiv (\chi'_i, \mu^2_i) \circ (\chi_i, \mu^1_i) = Gr(\chi') \circ Gr(\chi).$$

Moreover, the Gr-image of the identity G-bundle morphism is the identity G⋉$\mathscr{M}$-bundle morphism,

$$Gr({\rm Id}_{\mathcal{P}_{\rm G}}) = (e, \mu^1_i) \equiv ({\rm Id}_{\mu^1_i}) = {\rm Id}_{Gr(\mathcal{P}_{\rm G})}.$$

Thus, the mapping Gr does, indeed, define a functor

$$Gr \;:\; {\rm G}\text{-}\mathfrak{Bun}(\Sigma \,\|\, \mathscr{M}) \to {\rm G}\ltimes\mathscr{M}\text{-}\mathfrak{Bun}(\Sigma).$$

We shall next demonstrate that the latter functor is an equivalence.

We begin by showing that Gr is essentially surjective. Take a G⋉$\mathscr{M}$-bundle $\mathcal{P}_{{\rm G}\ltimes\mathscr{M}} = (\mathsf{P}_{{\rm G}\ltimes\mathscr{M}}, \Sigma, \pi_{\mathsf{P}_{{\rm G}\ltimes\mathscr{M}}}, \mu_{\mathsf{P}_{{\rm G}\ltimes\mathscr{M}}}, \rho_{\mathsf{P}_{{\rm G}\ltimes\mathscr{M}}})$ with local momenta $\mu_i : \Sigma_i \to \mathscr{M}$ and transition maps $\gamma_{ij} : \Sigma_{ij} \to {\rm G} \times \mathscr{M}$. The latter decompose as

$$\gamma_{ij} = (\gamma^{\rm G}_{ij}, \gamma^{\mathscr{M}}_{ij}), \qquad \gamma^{\mathscr{X}}_{ij} \;:\; \Sigma_{ij} \to \mathscr{X}, \quad \mathscr{X} \in \{\mathscr{M}, {\rm G}\}, \tag{8.31}$$

and identities (i) of Definition 8.34 yield, for $\sigma \in \Sigma_{ij}$,

$$\gamma_{ij}^{\mathscr{M}}(\sigma) = s \circ \gamma_{ij}(\sigma) = \mu_j(\sigma)\,, \qquad \gamma_{ij}^{\mathrm{G}}(\sigma).\mu_j(\sigma) = \gamma_{ij}^{\mathrm{G}}(\sigma).\gamma_{ij}^{\mathscr{M}}(\sigma) = t \circ \gamma_{ij}(\sigma) = \mu_i(\sigma)\,,$$

$$\big(\gamma_{ii}^{\mathrm{G}}(\sigma), \mu_i(\sigma)\big) = \big(\gamma_{ii}^{\mathrm{G}}(\sigma), \gamma_{ii}^{\mathscr{M}}(\sigma)\big) \equiv \gamma_{ii}(\sigma) = \mathrm{Id}_{\mu_i(\sigma)} = (e, \mu_i(\sigma))\,.$$

Identity (ii) of the same definition now implies

$$\big(\gamma_{ji}^{\mathrm{G}}(\sigma), \mu_i(\sigma)\big) = \gamma_{ji}(\sigma) = \big(\gamma_{ij}^{\mathrm{G}}(\sigma), \mu_j(\sigma)\big)^{-1} = \big(\gamma_{ij}^{\mathrm{G}}(\sigma)^{-1}, \gamma_{ij}^{\mathrm{G}}(\sigma).\mu_j(\sigma)\big) = \big(\gamma_{ij}^{\mathrm{G}}(\sigma)^{-1}, \mu_i(\sigma)\big)\,,$$

and identity (iii) transcribes as

$$\begin{aligned}\big(\gamma_{ik}^{\mathrm{G}}(\sigma), \mu_k(\sigma)\big) &= \gamma_{ik}(\sigma) = \gamma_{ij}(\sigma) \circ \gamma_{jk}(\sigma) = \big(\gamma_{ij}^{\mathrm{G}}(\sigma), \mu_j(\sigma)\big) \circ \big(\gamma_{jk}^{\mathrm{G}}(\sigma), \mu_k(\sigma)\big) \\ &= \big(\gamma_{ij}^{\mathrm{G}}(\sigma), \gamma_{jk}^{\mathrm{G}}(\sigma).\mu_k(\sigma)\big) \circ \big(\gamma_{jk}^{\mathrm{G}}(\sigma), \mu_k(\sigma)\big) = \big(\gamma_{ij}^{\mathrm{G}}(\sigma) \cdot \gamma_{jk}^{\mathrm{G}}(\sigma), \mu_k(\sigma)\big)\end{aligned}$$

for any $\sigma \in \Sigma_{ijk}$. Thus, altogether, the local bundle data consist of the smooth functions

$$\mu_i \;:\; \Sigma \to \mathscr{M}\,, \qquad g_{ij} \coloneqq \gamma_{ij}^{\mathrm{G}} \;:\; \Sigma_{ij} \to \mathrm{G}$$

with the following properties

$$\mu_i(\sigma) = g_{ij}(\sigma).\mu_j(\sigma)\,,$$

$$g_{ik}(\sigma) = g_{ij}(\sigma) \cdot g_{jk}(\sigma)\,, \qquad g_{ji}(\sigma) = g_{ij}(\sigma)^{-1}\,, \qquad g_{ii}(\sigma) = e\,.$$

Using the data g_{ij}, we obtain a principal G-bundle $\mathcal{P}_{\mathcal{O}_\Sigma} = (\mathsf{P}_{\mathcal{O}_\Sigma}, \Sigma, \pi_{\mathsf{P}_{\mathcal{O}_\Sigma}}, \rho_{\mathsf{P}_{\mathcal{O}_\Sigma}})$ via the standard clutching construction. Its total space is

$$\mathsf{P}_{\mathcal{O}_\Sigma} \coloneqq \bigsqcup_{i \in \mathscr{I}} (\Sigma_i \times \mathrm{G}) / \sim_{(g_{ij})}\,,$$

with the equivalence relation defined as

$$(i, \sigma_i, g_i) \sim_{(g_{ij})} (j, \sigma_j, g_j) \quad \Leftrightarrow \quad (\;\sigma_j = \sigma_i \in \Sigma_{ij} \quad \wedge \quad g_i = g_{ij}(\sigma_i) \cdot g_j\;)\,.$$

The projection to the base Σ reads

$$\pi_{\mathsf{P}_{\mathcal{O}_\Sigma}} \;:\; \mathsf{P}_{\mathcal{O}_\Sigma} \to \Sigma \;:\; \big[(i, \sigma, g)\big] \mapsto \sigma\,,$$

and the right G-action is given by the formula

$$\rho_{\mathsf{P}_{\mathcal{O}_\Sigma}} \;:\; \mathsf{P}_{\mathcal{O}_\Sigma} \times \mathrm{G} \to \mathsf{P}_{\mathrm{G}} \;:\; \big([(i, \sigma, g)], h\big) \mapsto [(i, \sigma, g \cdot h)]\,.$$

The latter is manifestly fibre-preserving, free and transitive.

Local trivialisations of $\mathcal{P}_{\mathcal{O}_\Sigma}$ are given by the maps

$$\tau_i \;:\; \pi_{\mathsf{P}_{\mathcal{O}_\Sigma}}^{-1}(\Sigma_i) \to \Sigma_i \times \mathrm{G} \;:\; [(i, \sigma, g)] \mapsto (\sigma, g)$$

with inverses

$$\tau_i^{-1} \;:\; \Sigma_i \times \mathrm{G} \to \pi_{\mathsf{P}_{\mathcal{O}_\Sigma}}^{-1}(\Sigma_i) \;:\; (\sigma, g) \mapsto [(i, \sigma, g)]$$

that have the desired G-equivariance property

$$\tau_i^{-1}(\sigma, g) = [(i, \sigma, e \cdot g)] = [(i, \sigma, e)]\,.g = \tau_i^{-1}(\sigma, e).g$$

and hence, in particular, satisfy the gluing relations

$$\tau_i^{-1}(\sigma, e) = [(i, \sigma, e)] = \big[\big(j, \sigma, g_{ji}(\sigma)\big)\big] = [(j, \sigma, e)].g_{ji}(\sigma) = \tau_j^{-1}(\sigma, e).g_{ji}(\sigma)\,.$$

The associated transition maps read

$$\tau_{ij} \coloneqq \tau_i \circ \tau_j^{-1} \;:\; (\sigma, g) \mapsto [(j, \sigma, g)] = \big[\big(i, \sigma, g_{ij}(\sigma) \cdot g\big)\big] \mapsto \big(\sigma, g_{ij}(\sigma) \cdot g\big)\,.$$

The τ_i in conjunction with the μ_i give rise to *global* sections of the associated bundle $\mathsf{P}_{\mathcal{O}_\Sigma} \times_{\mathrm{G}} \mathscr{M} \to \Sigma$, with the total space given by the smooth quotient $(\mathsf{P}_{\mathcal{O}_\Sigma} \times \mathscr{M})/\mathrm{G}$ with respect to the (right) diagonal G-action of Eq. (8.15). Indeed, write

$$\eta_i \;:\; \Sigma_i \to (\mathsf{P}_{\mathcal{O}_\Sigma} \times \mathscr{M})/\mathrm{G} \;:\; \sigma \mapsto \big[\big(\tau_i^{-1}(\sigma, e), \mu_i(\sigma)\big)\big]\,.$$

We readily check that the η_i compose a global section as for an arbitrary $\sigma \in \Sigma_{ij}$,

$$\begin{aligned}\eta_j(\sigma) = \big[\big(\tau_j^{-1}(\sigma, e), \mu_j(\sigma)\big)\big] &= \big[\big(\tau_i^{-1}(\sigma, e).g_{ij}(\sigma), \mu_j(\sigma)\big)\big] = \big[\big(\tau_i^{-1}(\sigma, e), g_{ij}(\sigma).\mu_j(\sigma)\big)\big] \\ &= \big[\big(\tau_i^{-1}(\sigma, e), \mu_i(\sigma)\big)\big] = \eta_i(\sigma)\,.\end{aligned}$$

The above construction yields a map

$$\daleth\ :\ \mathrm{Ob}\,\mathrm{G}\ltimes\mathscr{M}\text{-}\mathfrak{Bun}(\Sigma) \to \mathrm{Ob}\,\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma\,\|\,\mathscr{M})\ :\ \mathcal{P}_{\mathrm{G}\ltimes\mathscr{M}} \mapsto \mathcal{P}_{\mathcal{O}_\Sigma}\,,$$

and we readily check that

$$Gr\circ\daleth(\mathcal{P}_{\mathrm{G}\ltimes\mathscr{M}}) \cong \mathcal{P}_{\mathrm{G}\ltimes\mathscr{M}}\,,$$

with the isomorphism determined by local trivialisations of $\mathcal{P}_{\mathrm{G}\ltimes\mathscr{M}}$ as described in Proposition 8.36. This proves that Gr is essentially surjective, as claimed.

In the next step, we show that Gr is full by explicitly constructing a counterpart of $\daleth$ acting on morphisms, to be denoted by the same symbol. Here, we consider a pair of principal $\mathrm{G}\ltimes\mathscr{M}$-bundles $\mathcal{P}^A_{\mathrm{G}\ltimes\mathscr{M}},\ A\in\{1,2\}$ over Σ with the respective local data $(\mathcal{O}_\Sigma,\mu^A_i,\gamma^A_{ij}\mid i,j\in\mathscr{I})$ associated with an open cover introduced before. Following Proposition 8.38, we take the data to be related as

$$\mu^1_i = s\circ\theta_i\,,\qquad\qquad \mu^2_i = t\circ\theta_i\,,$$

and, for any $\sigma\in\Sigma_{ij}$,

$$\gamma^2_{ij}(\sigma) = \theta_i(\sigma)\circ\gamma^1_{ij}(\sigma)\circ\theta_j(\sigma)^{-1}$$

by a collection $(\theta_i)_{i\in\mathscr{I}}$ of locally smooth maps $\theta_i:\Sigma_i\to\mathrm{Mor}\,(\mathrm{G}\ltimes\mathscr{M})$. Taking into account the specific form of the source and target maps of $\mathrm{G}\ltimes\mathscr{M}$, we may write the θ_i in the component form

$$\theta_i = (\xi_i,\mu^1_i)\,,$$

with $\xi_i:\Sigma_i\to\mathrm{G}$ chosen such that, for all $\sigma\in\Sigma_i$,

$$\xi_i(\sigma).\mu^1_i(\sigma) = t\circ\theta_i(\sigma) = \mu^2_i(\sigma)\,.$$

Writing out the γ^A_{ij} in components as in Eq. (8.31), we then find over $\Sigma_{ij}\ni\sigma$,

$$\begin{aligned}\big(g^2_{ij}(\sigma),\mu^2_j(\sigma)\big) &\equiv \gamma^2_{ij}(\sigma) = \theta_i(\sigma)\circ\gamma^1_{ij}(\sigma)\circ\theta_j(\sigma)^{-1}\\ &= \big(\xi_i(\sigma),\mu^1_i(\sigma)\big)\circ\big(g^1_{ij}(\sigma),\mu^1_j(\sigma)\big)\circ\big(\xi_j(\sigma)^{-1},\xi_j(\sigma).\mu^1_j(\sigma)\big)\\ &= \big(\xi_i(\sigma)\cdot g^1_{ij}(\sigma)\cdot\xi_j(\sigma)^{-1},\xi_j(\sigma).\mu^1_j(\sigma)\big)\,,\end{aligned}$$

whence we infer that the local data of the bundles $\daleth(\mathcal{P}^1_{\mathrm{G}\ltimes\mathscr{M}})$ and $\daleth(\mathcal{P}^2_{\mathrm{G}\ltimes\mathscr{M}})$ are related by a G-bundle (iso)morphism

$$\daleth(\Theta)\ :\ \daleth(\mathcal{P}^1_{\mathrm{G}\ltimes\mathscr{M}}) \xrightarrow{\cong} \daleth(\mathcal{P}^2_{\mathrm{G}\ltimes\mathscr{M}})$$

with local data (ξ_i) (that automatically belongs to $\mathrm{Hom}_{\mathrm{G}\text{-}\mathfrak{Bun}(\Sigma)}\big(\daleth(\mathcal{P}^1_{\mathrm{G}\ltimes\mathscr{M}}),\daleth(\mathcal{P}^2_{\mathrm{G}\ltimes\mathscr{M}})\big)$). It is now a matter of a simple check to see that

$$Gr\circ\daleth(\Theta) = \Theta\,.$$

The last property of Gr to be substantiated is its faithfulness. This one follows immediately from the construction of the functor. Indeed, different G-bundle isomorphisms would necessarily have different local data that would – in turn – yield different $\mathrm{G}\ltimes\mathscr{M}$-bundle isomorphisms under Gr. Thus, we have established that the functor Gr is an equivalence of categories, which concludes the proof of the theorem. □

Remark 8.42. The above theorem is an extension of the statement of a one-to-one correspondence between objects of the two categories worked out in Ref. [Ros04a, Sec. 3.3.3].

The last theorem offers a most natural explanation of the appearance of the tangent algebroid of the action groupoid $\mathrm{G}_\sigma\ltimes\mathscr{F}$ in the analysis of the rigid symmetries of the σ-model that admit gauging. On top of that, it gives rise to a simple local presentation of field configurations of the gauged σ-model on the purely geometric level, *i.e.* in the setting in which the presence of the gauge field on the world-sheet and that of the metric and gerbe-theoretic structure on the target space has been forgotten. In this presentation, sketched in Figure 1 in the simplest case of a mono-phase σ-model, elements of an open cover of the world-sheet are embedded smoothly into the target space using local momenta of the principal $\mathrm{G}_\sigma\ltimes\mathscr{F}$-bundle Gr-dual to the principal G_σ-bundle of the gauged σ-model. This is done in such a manner that images, under the respective local momenta, of points from double intersections of elements of the open cover are related by arrows from the morphism set $\mathrm{G}_\sigma\times\mathscr{F}$ of the action groupoid determined by the appropriate transition maps of the principal G_σ-bundle. The picture thus obtained is largely reminiscent of the well-established idea of realising field configurations

of the σ-model on the orbit space of the action of a group on the target space of a parent σ-model through patchwise smooth field configurations of the parent σ-model, in which field discontinuities that occur upon passing between neighbouring patches are determined by the action of the group that is being gauged. Of course, for this idea to be applicable, one would have to take into account the extra structure, both on the world-sheet[19] and on the target space, that enters the definition of the gauged σ-model. Nevertheless, even in its present over-simplified form, it does provide us with qualitative insights into the local structure of the gauged σ-model, and that with direct reference to the algebroidal structure discovered earlier on the set of infinitesimal symmetries under gauging. We shall take up this newly established intuition in the next section and combine it, along the lines of Refs. [RS09, Sec. 2] and [FFRS09, Sec. 3], with the concept of a duality defect of Ref. [Sus11, Sec. 3] with view to obtaining a world-sheet definition of a field configuration of the gauged σ-model (in the presence of the full-fledged differential-geometric structure on the world-sheet and on the target space) locally twisted by the symmetry group under gauging. Remarkably enough, as a byproduct of our analysis, we find a novel field-theoretic interpretation of the large gauge anomaly. But even prior to such refinement, the theorem clearly demonstrates, on purely geometric grounds[20], the necessity of having gauge fields of *arbitrary* topology coupled to the string background of the parent σ-model (with the target space $\mathscr{F}$) for a *complete* formulation of the gauged resp. coset σ-model, taking into account the existence of the G_σ-twisted sector.

8.3. **Topological gauge-symmetry defect networks and G_σ-equivariance.** Our hitherto careful investigation of the algebraic aspects of the passage from global symmetries of the multi-phase σ-model to their local counterparts has brought to the fore the rôle of the action groupoid $\mathrm{G}_\sigma \ltimes \mathscr{F}$ as the structure underlying symmetries of the gauged σ-model. Furthermore, it has led to the emergence of a suggestive local geometric picture of the latter field theory. In the remainder of this section, we want to formalise these observations in a manner consistent with the extra structure present on the world-sheet (the gauge field) and over the target space (the string background). The findings of the previous section suggest two directions in which we can develop the discussion of the gauged σ-model, to wit,

- a local implementation of the gauge symmetry through patchwise smooth network-field configurations with $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump discontinuities localised along (topological-)defect lines, forming an arbitrarily dense mesh as in Refs. [RS09, Sec. 2] and [FFRS09, Sec. 3];
- a systematic reconstruction of network-field configurations in the background of a *topologically non-trivial* gauge field through local trivialisation of the gauge bundle and subsequent application of the clutching construction using local transition maps.

Technically speaking, the two constructions are intimately related: Both entail splitting Σ into a collection of patches Σ_i through the embedding of an oriented graph Γ (a defect graph resp. a graph defining the triangulation of Σ subordinate to the open cover used in the local trivialisation) and pulling back data of local trivialisations of the geometric structure over the world-sheet (*i.e.* the gauge field coupled to the string background) to the patches, and data of local morphisms relating the trivialisations to the edges and vertices of the graph. Both impose consistency conditions on the data pulled back to the multi-valent vertices of the graph (associativity *etc.*). Finally, both require (local) extendibility of the local data (to ensure topologicality, a distinctive feature of a duality defect network, resp. to ensure independence of the construction of the arbitrary choices made in the trivialisation procedure). The sole formal difference between the two constructions consists in the choice of the gluing maps $\chi : \mathfrak{E}_\Gamma \sqcup \mathfrak{V}_\Gamma \to \mathrm{G}_\sigma$ (*cf.* Definition I.2.6), but that is readily accounted for: In the former case, one uses restrictions of globally defined (smooth) maps $\chi \in C^\infty(\Sigma, \mathrm{G}_\sigma)$; in the latter case, the construction of topologically non-trivial gauge bundles necessitates the use of locally smooth maps $\chi_{ij} \in C^\infty(\Sigma_{ij}, \mathrm{G}_\sigma)$ without global extensions, *cf.* the discussion closing the previous section. In the light of the structural affinity between the two constructions, and with view to keeping the discourse less cluttered with technical notation, we choose to present in detail only the first construction. Incidentally, this will enable us to give an explicit realisation of the abstract ideas outlined in Remark I.5.6, and – in so

[19] An extension of the equivalence between the category of principal G_σ-bundles gauging the target space of the σ-model and the category of principal bundles with the corresponding action groupoid to the setting *with connection* should be possible and relatively straightforward within the differential-geometric framework developed in Refs. [Mac87] and [SW09]. We hope to return to this issue in future work.

[20] In Refs. [GSW10, GSW12], the incorporation of topologically non-trivial gauge fields into a unified framework was motivated by purely field-theoretic arguments relying on inconclusive (in this respect) analyses of Refs. [SY89, SY90, Hor96, FSS96].

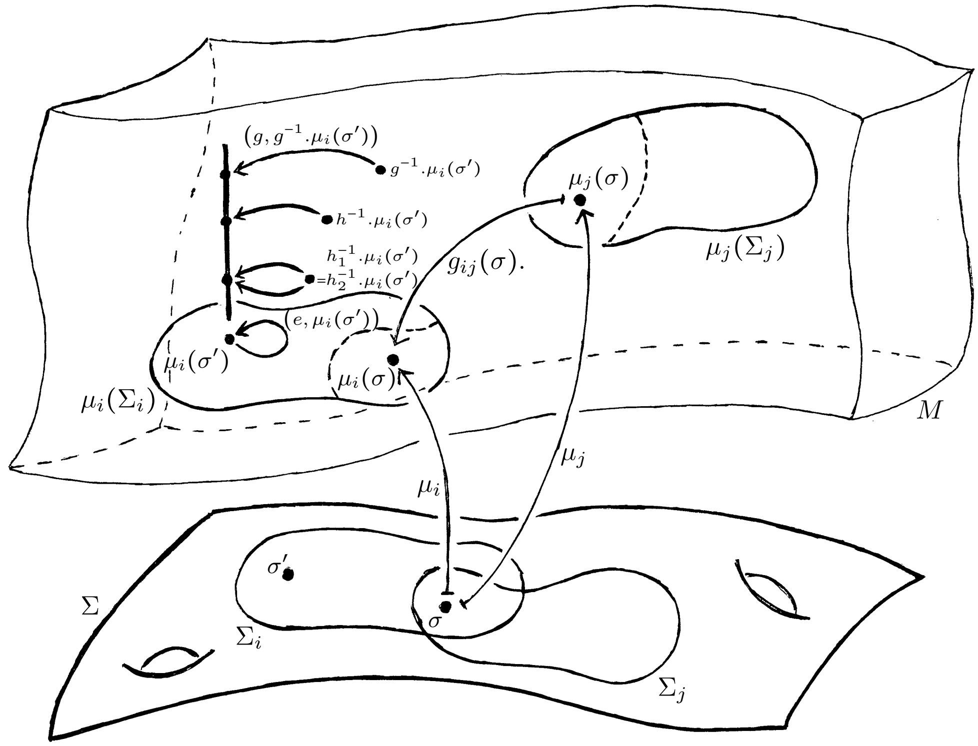

FIGURE 1. The principal $\mathrm{G}_\sigma\ltimes M$-bundle $Gr(\mathcal{P}_{\mathrm{G}_\sigma})$ over Σ in the (dual) local description of the gauged mono-phase σ-model. Open neighbourhoods $\Sigma_i \subset \Sigma$ are mapped into M by local momenta μ_i extracted from the definition of a global section of the associated bundle $\mathsf{P}_{\mathrm{G}_\sigma} \times_{\mathrm{G}_\sigma} M$. Points in the image of a double intersection Σ_{ij} of the neighbourhoods are related by the action of the transition map g_{ij} of the principal G_σ-bundle $\mathcal{P}_{\mathrm{G}_\sigma}$. Over each point $\mu_i(\sigma')$, there is an entire fibre of arrows from $\mathrm{G}_\sigma \times M$ ending at $\mu_i(\sigma')$. In addition to the complete information about the G_σ-orbit $\mathrm{G}_\sigma.\mu_i(\sigma)$, the fibre encodes information on the isotropy subgroup $\mathrm{G}_{\sigma\,\mu_i(\sigma)}$ (*cf.* the pair of arrows with a common source and target).

doing – will provide us with a new interpretation of the large gauge anomaly. Upon completing the presentation, we comment briefly on the application of the methods developed along the way in the second construction.

The embedding in the world-sheet of a defect network implementing the action of the gauge group on fields of the gauged *multi-phase* σ-model divides naturally into three stages. The first stage is restricted to a single phase of the theory. It consists in defining the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump bi-brane and ensuring that the associated (component) $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defects are topological and can be fused in an associative manner, leading to the emergence of a topological $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network. In the second stage, one ensures compatibility of the former definition with the structure of a conformal defect $\mathscr{D}_{\mathrm{A}}$ between phases of the gauged σ-model (assuming $\mathscr{D}_{\mathrm{A}}$ to be G_σ-symmetric) by defining a junction between $\mathscr{D}_{\mathrm{A}}$ and an arbitrary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect, and by requiring subsequently that the presence of $\mathscr{D}_{\mathrm{A}}$ do not destroy the crucial feature of topologicality of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network. The third and final stage of the construction boils down to securing topologicality in the presence of self-intersections of $\mathscr{D}_{\mathrm{A}}$. We shall now go step by step through the successive stages.

Let us start by taking into consideration a single phase of the gauged σ-model. The point of departure in our discussion is the following

Definition 8.43. Adopt the notation of Definitions I.2.6 and 8.2, and of Proposition 8.7. Given an arbitrary map $\chi \in C^\infty(\Sigma, \mathrm{G}_\sigma)$, the associated $C^\infty(\Sigma, \mathrm{G}_\sigma)$**-jump defect** $\mathscr{D}_\chi$ for the gauged σ-model of Eq. (8.1) is the one-dimensional locus $\ell \subset \Sigma$ (of the topology of a line segment or that of a circle) of

discontinuity of the lagrangean fields of the theory of the form

$$X_{|1}(p) = \chi(p).X_{|2}(p)\,, \qquad\qquad \mathrm{A}_{|1}(p) = {}^\chi\mathrm{A}_{|2}(p)\,, \qquad p \in \ell\,, \tag{8.32}$$

cf. Figure 2, carrying the data of the distinguished **(component)** $C^\infty(\Sigma, \mathrm{G}_\sigma)$**-jump bi-brane**, that is the $(L_\chi^*\mathcal{G}_{\chi\mathrm{A}}, \mathcal{G}_\mathrm{A})$-bi-brane

$$\mathcal{B}_\chi \coloneqq \left(\{\chi\} \times \Sigma \times M \equiv \Sigma \times M, L_\chi, \mathrm{id}_{\Sigma\times M}, \Upsilon_\chi, 0\right)\,,$$

written in terms of the gerbe 1-isomorphism

$$\Upsilon_\chi \coloneqq (\chi \times \mathrm{id}_M)^*\Upsilon \;:\; L_\chi^*\mathcal{G}_{\chi\mathrm{A}} \xrightarrow{\cong} \mathcal{G}_\mathrm{A}\,,$$

with

$$L_\chi \;:\; \Sigma \times \mathscr{F} \to \Sigma \times \mathscr{F} \;:\; (\sigma, m) \mapsto (\sigma, \chi(\sigma).m)\,.$$

The pair $\left(\xi_{|2}, \mathrm{A}_{|2}\right)$, with $\xi_{|2} \equiv (\mathrm{id}_\Sigma, X_{|2})$ are taken as the restriction of the field configuration of the (gauged) σ-model in the presence of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect $\mathscr{D}_\chi$ to the defect line ℓ. In keeping with the original notation of Ref. [RS09, Sec. 2], they are to be denoted as $(X, \mathrm{A})|_\ell$.

Taking a disjoint union over the gauge group of component $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump bi-branes associated with various maps $\chi \in C^\infty(\Sigma, \mathrm{G}_\sigma)$, we obtain the **(total)** $C^\infty(\Sigma, \mathrm{G}_\sigma)$**-jump bi-brane**

$$\mathcal{B}_{C^\infty(\Sigma,\mathrm{G}_\sigma)} = \bigsqcup_{\chi\in C^\infty(\Sigma,\mathrm{G}_\sigma)} \mathcal{B}_\chi\,.$$

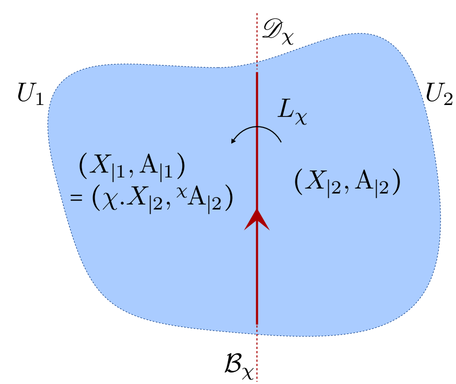

Figure 2. The $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect $\mathscr{D}_\chi$ associated with the mapping $\chi \in C^\infty(\Sigma, \mathrm{G}_\sigma)$ and carrying the data of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump bi-brane $\mathcal{B}_\chi$.

The physical relevance of the above definition stems from the following

Proposition 8.44. *The* $C^\infty(\Sigma, \mathrm{G}_\sigma)$*-jump defect* $\mathscr{D}_\chi$ *of Definition 8.43 is conformal in the sense of Ref.* [RS09, Sec. 2.9].

Proof. As shown in Ref. [RS09, Sec. 2.9], conformality of a world-sheet defect is implied by the Defect Gluing Condition (I.2.8) being satisfied by the corresponding circle-field configuration. Hence, it suffices to verify the appropriate DGC for $\mathscr{D}_\chi$, obtained as the term in the variation of the action functional (8.1) localised at the defect line. In order to simplify matters further[21], take a cohomologically trivial target

$$\mathcal{M} = (M, \mathrm{g}, \mathcal{G})\,, \qquad \mathcal{G} \coloneqq I_\mathrm{B}\,, \tag{8.33}$$

endowed with a cohomologically trivial G_σ-equivariant structure (*cf.* Ref. [GSW12, Def. 8.1])

$$(\Upsilon, \gamma) \coloneqq (J_E, f)\,, \qquad (E, f) \in \Omega^1(\mathrm{G}_\sigma \times M, \mathbb{R}) \times C^\infty\left(\mathrm{G}_\sigma^2 \times M, \mathbb{R}\right)\,. \tag{8.34}$$

[21] A lengthy but otherwise completely straightforward analysis free of such simplifying assumptions can readily be carried out along the lines of Ref. [RS09, App. A.2].

Use the adapted world-sheet coordinates $(\sigma^1, \sigma^2) \equiv (t, \varphi)$ in which the defect line ℓ is the locus of the equation $t = 0$ and the right-handed basis of $\mathsf{T}_p\Sigma$ considered in Definition I.2.6 is given by $(\partial_t, \partial_\varphi)$. Given a variation $X^\mu \mapsto X^\mu + \mathscr{V}^\mu$, $\mathscr{V} \in \Gamma(\mathsf{T}M)$ the DGC reads

$$\begin{aligned}
\mathrm{DGC}(X;\mathrm{A})(\varphi) \quad := \quad & \mathrm{g}_{\mu\nu}\left(\chi.X_{|2}(0,\varphi)\right)\left(D_{{}^\chi\mathrm{A}_{|2}}(\chi.X_{|2})^\mu\right)_t(0,\varphi)\,(\ell_{\chi(0,\varphi)\,*}\mathscr{V})^\nu\left(X_{|2}(0,\varphi)\right) \\
& -\mathrm{g}_{\mu\nu}\left(X_{|2}(0,\varphi)\right)\left(D_{\mathrm{A}_{|2}}(X_{|2})^\mu\right)_t(0,\varphi)\,\mathscr{V}^\nu\left(X_{|2}(0,\varphi)\right) \\
& +\kappa_{A\,\mu}\left(\chi.X_{|2}(0,\varphi)\right)\left({}^\chi\mathrm{A}^A_{|2}\right)_\varphi(\ell_{\chi(0,\varphi)\,*}\mathscr{V})^\mu\left(X_{|2}(0,\varphi)\right) \\
& -\kappa_{A\,\mu}\left(X_{|2}(0,\varphi)\right)\left(\mathrm{A}^A_{|2}\right)_\varphi\mathscr{V}^\mu\left(X_{|2}(0,\varphi)\right) \\
& +2\mathrm{B}_{\mu\nu}\left(\chi.X_{|2}(0,\varphi)\right)(\ell_{\chi(0,\varphi)\,*}\mathscr{V})^\mu\left(X_{|2}(0,\varphi)\right)\partial_\varphi\left(\chi.X_{|2}\right)^\nu(0,\varphi) \\
& -2\mathrm{B}_{\mu\nu}\left(X_{|2}(0,\varphi)\right)\mathscr{V}^\mu\left(X_{|2}(0,\varphi)\right)\partial_\varphi X^\nu_{|2}(0,\varphi) \\
& +\xi_{|2\,*}\partial_\varphi \lrcorner\, \mathscr{V}\left(X_{|2}(0,\varphi)\right) \lrcorner\, \mathsf{d}E_\chi\left((0,\varphi), X_{|2}(0,\varphi)\right)
\end{aligned}$$

with

$$E_\chi := (\chi \times \mathrm{id}_M)^* E\,.$$

It is our task to show that the DGC vanishes identically. Its first two terms cancel out due to the assumed G_σ-invariance of the target-space metric (recall the tensorial transformation law for the covariant derivative, *cf.* Ref. [GSW12, Eq. (3.13)]). Taking into account the G_σ-equivariance of κ, *cf.* Eq. (8.9), in the integrated form

$${}^M\ell^*_{\chi(\sigma)}\kappa(X) = \kappa\left(\mathrm{Ad}_{\chi(\sigma)^{-1}}X\right)\,, \qquad X \in \mathfrak{g}_\sigma\,,$$

in conjunction with the defining formula

$$\mathsf{d}E_\chi(\sigma, m) = \mathrm{B}(m) - \mathrm{e}^{-\overline{\chi^*\theta_L(\sigma)}} \cdot {}^M\ell^*_{\chi(\sigma)}\mathrm{B}(m) + \rho_{\chi^*\theta_L}(\sigma, m)$$

that uses the notation of Ref. [GSW12, Conv. 2.11], we reduce the DGC to the form

$$\begin{aligned}
\mathrm{DGC}(X;\mathrm{A})(\varphi) \quad = \quad & -\kappa_{A\,\mu}\left(X_{|2}(0,\varphi)\right)\left(\chi^{-1}\,\partial_\varphi\chi\right)^A(0,\varphi)\,\mathscr{V}^\mu\left(X_{|2}(0,\varphi)\right) \\
& +2\mathrm{B}_{\mu\nu}\left(\chi.X_{|2}(0,\varphi)\right)(\ell_{\chi(0,\varphi)\,*}\mathscr{V})^\mu\left(X_{|2}(0,\varphi)\right)\partial_\varphi\left(\chi.X_{|2}\right)^\nu(0,\varphi) \\
& -2\mathrm{B}_{\mu\nu}\left(X_{|2}(0,\varphi)\right)\mathscr{V}^\mu\left(X_{|2}(0,\varphi)\right)\partial_\varphi X^\nu_{|2}(0,\varphi) \\
& +X_{|2\,*}\partial_\varphi \lrcorner\, \mathscr{V} \lrcorner\left(\mathrm{B} - {}^M\ell^*_{\chi(0,\varphi)}\mathrm{B}\right)\left(X_{|2}(0,\varphi)\right) \\
& -\left(\chi^{-1}\,\partial_\varphi\chi\right)^A(0,\varphi)\left(\mathscr{V} \lrcorner\, {}^M\mathscr{K}_A \lrcorner\, {}^M\ell^*_{\chi(0,\varphi)}\mathrm{B} - \mathscr{V} \lrcorner\, \kappa_A\right)\left(X_{|2}(0,\varphi)\right)
\end{aligned}$$

It is now evident that the DGC vanishes identically, *cf.* Ref. [GSW12, Eq. (2.36)].

We conclude that the existence of Υ ensures the existence of an *element-wise* realisation of $C^\infty(\Sigma, \mathrm{G}_\sigma)$ on extended targets through equivalences (*i.e.* it maps a target to a physically equivalent one). □

The study, initiated in Ref. [RS09] and carried out at length in Ref. [Sus11], of the correspondence between conformal defects and dualities of the σ-model (the latter being understood in the sense of Definition I.4.7) has singled out the topological defects of Definition I.4.3 as natural candidates for world-sheet representatives of the said dualities. This conforms with predictions of various alternative approaches to the CFT of the σ-model, including those of the categorial quantisation scheme reported in Refs. [FFRS04, FFRS07]. We are thus led to enquire as to the topologicality of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect. The answer to this question is given in the following

Proposition 8.45. *The circle-field configuration (understood in the sense of Ref.* [RS09, Sec. 2.4]*) for the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect $\mathscr{D}_\chi$ of Definition 8.43 is extendible, and so the defect is topological in the sense of Ref.* [RS09, Sec. 2.9].

Proof. An extension

$$\widehat{\xi} := (\mathrm{id}_U, \widehat{X}) \;:\; U \to U \times M$$

of a circle-field configuration $(\xi\,|\,\Gamma)$ on a world-sheet Σ with an embedded (oriented) circular defect line $\Gamma \cong \mathbb{S}^1$ that carries the data of $\mathscr{D}_\chi$ to a tubular neighbourhood U of Γ within Σ takes the form

$$\widehat{X}(\sigma) = \begin{cases} L_{\mathrm{Inv}\circ\chi} \circ X(\sigma) & \text{if } \sigma \in U_1 \\ \\ X(\sigma) & \text{if } \sigma \in U_2 \end{cases} .$$

The extension of the original configuration A assigned to $\mathscr{D}_\chi$ as in Definition 8.43 reads

$$\widehat{\mathrm{A}}(\sigma) = \begin{cases} {}^{\mathrm{Inv}\circ\chi}\mathrm{A}(\sigma) & \text{if } \sigma \in U_1 \\ \\ \mathrm{A}(\sigma) & \text{if } \sigma \in U_2 \end{cases} .$$

Adducing the very same arguments as in the proof of the vanishing of $\mathrm{DGC}(X;\mathrm{A})$, we convince ourselves that the above extension of the defect field configuration $(X,\mathrm{A})|_\ell$ satisfies Eq. (2.113) of Ref. [RS09], from which we infer that the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect is extendible, and hence – by the arguments of Ref. [RS09, Sec. 2.9] – (off-shell) topological. □

It is to be stressed that the existence of a $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect is a straightforward consequence of the assumed $C^\infty(\Sigma,\mathrm{G}_\sigma)$-invariance of the gauged σ-model in the presence of the topologically trivial gauge field, by which we mean that it does not call for any additional structure beyond the one required for the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-invariance, *cf.* Proposition 8.7. On the other hand, from the arguments presented in the Introduction to Ref. [RS09], we infer that the presence of defects in a self-consistent quantum CFT unavoidably leads to the emergence of defect junctions at which the convergent defects undergo fusion. As seen from the world-sheet perspective, the latter is to be understood as a relation between the limiting values attained by the defect embedding maps together with a 2-isomorphism trivialising a (horizontal) composition of the pullbacks of the defect 1-isomorphisms to the inter-bi-brane world-volume in which the defect junction is embedded, both following the scheme detailed in Ref. [RS09, Sec. 2.5]. Thus, internal consistency of the field theory in hand is contingent upon the existence of the above-mentioned fusion 2-isomorphism.

While there is no *a priori* relation between the inter-bi-brane world-volume and the components of the bi-brane world-volume into which the convergent defect lines are mapped, or between world-volumes of inter-bi-branes corresponding to junctions of different valence, the study of the structure of inter-bi-branes in specific situations in which the relevant defects implement the action of a symmetry group of the σ-model (such as, *e.g.*, the $Z(\mathrm{G})$-jump defects of the WZW model dealt with in Ref. [RS09], or the more general maximally symmetric defects of the same model analysed in Refs. [RS11, RS12] and [GSW12, Sec. 5]) indicates that a distinguished form of a string background is favoured in such circumstances, to wit, a string background with induction. This concept was introduced in Ref. [RS09, Sec. 2.8] and further elaborated in Ref. [Sus11, Rem. 5.6]. Its basis is the reconstruction of an elementary (trivalent) inter-bi-brane that we give in

Definition 8.46. Adopt the notation of Definitions I.2.6 and 8.43, and of Propositions 8.7 and 8.10. Given arbitrary maps $\chi_1,\chi_2 \in C^\infty(\Sigma,\mathrm{G}_\sigma)$, the associated **elementary $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect junction** $\mathscr{J}_{\chi_1,\chi_2}$ for the gauged σ-model of Eq. (8.1) is the point $\jmath_{(3)} \subset \Sigma$ of convergence of a triple of $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defects $\mathscr{D}_{\chi_1},\mathscr{D}_{\chi_2}$ and $\mathscr{D}_{\chi_1\cdot\chi_2}$ of the type depicted in Figure 3, carrying the data of the **(component) elementary $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump inter-bi-brane**

$$\mathcal{J}_{\chi_1,\chi_2} := \mathcal{B}_{\chi_1}\left(\{(\chi_1,\chi_2)\}\times\Sigma\times M \equiv \Sigma\times M;\pi_3^{1,2},\pi_3^{2,3},\pi_3^{3,1};\gamma_{\chi_1,\chi_2}\right),$$

including the inter-bi-brane maps

$$\pi_3^{1,2} \quad : \quad \{(\chi_1,\chi_2)\}\times\Sigma\times M \to \{\chi_1\}\times\Sigma\times M \ : \ (\chi_1,\chi_2,\sigma,m) \mapsto (\chi_1,\sigma,\chi_2(\sigma).m)\,,$$

$$\pi_3^{2,3} \quad : \quad \{(\chi_1,\chi_2)\}\times\Sigma\times M \to \{\chi_2\}\times\Sigma\times M \ : \ (\chi_1,\chi_2,\sigma,m) \mapsto (\chi_2,\sigma,m)\,,$$

$$\pi_3^{3,1} \quad : \quad \{(\chi_1,\chi_2)\}\times\Sigma\times M \to \{\chi_1\cdot\chi_2\}\times\Sigma\times M \ : \ (\chi_1,\chi_2,\sigma,m) \mapsto (\chi_1\cdot\chi_2,\sigma,m)$$

and the 2-isomorphism

$$\gamma_{\chi_1,\chi_2} := ((\chi_1,\chi_2)\times\mathrm{id}_M)^*\,\gamma \ : \ (\Upsilon_{\chi_2}\otimes\mathrm{Id})\circ L_{\chi_2}^*\Upsilon_{\chi_1} \xRightarrow{\cong} \Upsilon_{\chi_1\cdot\chi_2}\,.$$

Taking a disjoint union over the gauge group of component elementary $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump inter-bi-branes associated with various maps $\chi_1,\chi_2\in C^\infty(\Sigma,\mathrm{G}_\sigma)$, we obtain the **(total) elementary $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump inter-bi-brane**

$$\mathcal{J}^{++-}_{C^\infty(\Sigma,\mathrm{G}_\sigma)}:=\bigsqcup_{\chi_1,\chi_2\in C^\infty(\Sigma,\mathrm{G}_\sigma)}\mathcal{J}_{\chi_1,\chi_2}\,.$$

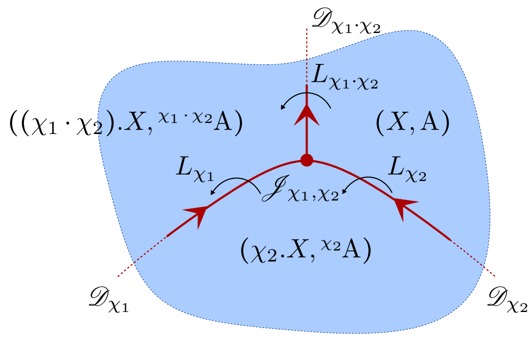

Figure 3. A trivalent junction $\mathscr{J}_{\chi_1,\chi_2}$ of the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defects: the two incoming ones, $\mathscr{D}_{\chi_1}$ and $\mathscr{D}_{\chi_2}$, and the outgoing product defect $\mathscr{D}_{\chi_1\cdot\chi_2}$.

Remark 8.47. Note that the component elementary $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump inter-bi-brane can be identified, by a slight abuse of the notation, with the product of bi-branes fibred over the target space in terms of the bi-brane maps,

$$\mathcal{J}_{\chi_1,\chi_2}=\mathcal{B}_{\chi_1}\,{}_{\mathrm{id}_{\Sigma\times M}}\!\times_{L_{\chi_2}}\mathcal{B}_{\chi_2}\,.$$

Under this identification, the inter-bi-brane maps become the canonical projections pr_1, pr_2 and $(\mathrm{m}\circ(\mathrm{pr}_1\circ\mathrm{pr}_1,\mathrm{pr}_1\circ\mathrm{pr}_2),(\mathrm{pr}_2,\mathrm{pr}_3)\circ\mathrm{pr}_2)$, respectively.

A distinctive feature of string backgrounds with induction is the extendibility of the associated network-field configurations *in the presence of defect junctions*. The feature allows to translate defect junctions along defect lines without changing the value of the σ-model action functional. In the present setting, we find

Proposition 8.48. *The network-field configuration (understood in the sense of Ref.* [RS09, Sec. 2.6]*) for a graph of the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defects of Definition 8.43 with at most trivalent junctions, as described in Definition 8.46, is extendible, and so the defect defined by the graph is topological in the sense of Ref.* [RS09, Sec. 2.9]*.*

Proof. We describe in full detail the extension of the network-field configuration in the vicinity of the defect junction drawn on the left-hand side of Figure 4, and study the effect of the local homotopic deformation of the defect quiver, using the extension, on the value of the σ-model action functional, *cf.* Ref. [RS09, App. A.3]. Extension of our considerations to generic homotopy moves of trivalent $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect junctions within the world-sheet is straightforward and therefore left as an exercise to the reader.
We define

$$\widehat{\xi}_{\ell_1}:=\big(\mathrm{id}_\Sigma,\widehat{X}_{\ell_1}\big)\quad:\quad\triangle\to\{\chi_1\}\times\triangle\times M\;:\;\sigma\mapsto\begin{cases}\big(\sigma,\chi_1(\sigma)^{-1}.X(\sigma)\big) & \text{if }\sigma\notin\ell_1\\[1em] (\sigma,X(\sigma)) & \text{if }\sigma\in\ell_1\end{cases}\,,$$

$$\widehat{\xi}_{\upsilon_1}:=\big(\mathrm{id}_\Sigma,\widehat{X}_{\upsilon_1}\big)\quad:\quad\ell_3\to\{(\chi_1,\chi_2)\}\times\ell_3\times M\;:\;\sigma\mapsto(\sigma,X(\sigma))\,.$$

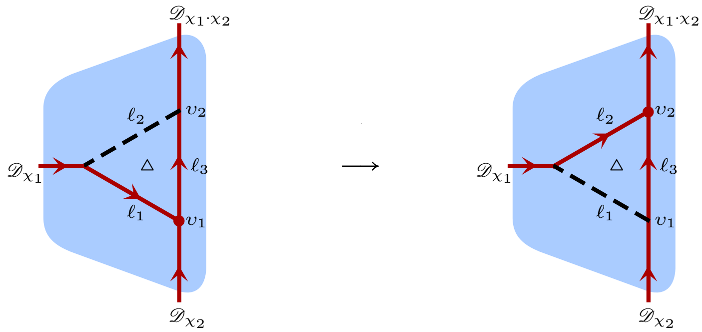

FIGURE 4. A homotopic displacement of a trivalent vertex of a $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect network along a defect line.

Assuming, as previously, cohomological triviality of the various gerbe-theoretic structures involved, we can calculate the difference between the values attained by the action functional on the two network-field configurations from Figure 4, of which the right one, $(\widetilde{X}\,|\,\widetilde{\Gamma})$, defined for $\widetilde{\Gamma}$ resulting from the homotopic displacement of the defect junction, is determined by the above extension as

$$\widetilde{X}|_{\Sigma\smallsetminus\triangle}=X|_{\Sigma\smallsetminus\triangle}\,,\qquad \widetilde{X}|_{\triangle\smallsetminus\ell_3}=\widehat{X}_{\ell_1}|_{\triangle\smallsetminus\ell_3}\,,\qquad \widetilde{X}|_{\ell_3}=\widehat{X}_{v_1}\,.$$

Completing the definition of the new configuration by redefining the gauge field in an obvious manner (it is understood that the limiting values attained by the gauge field on either side of a defect line are determined by the field's smooth functional dependence on the point in the bulk, as specified below),

$$\widetilde{\mathrm{A}}|_{\Sigma\smallsetminus\triangle}=\mathrm{A}|_{\Sigma\smallsetminus\triangle}\,,\qquad \widetilde{\mathrm{A}}|_{\triangle\smallsetminus\partial\triangle}={}^{\mathrm{Inv}\circ\chi_1}\mathrm{A}|_{\triangle\smallsetminus\partial\triangle}\,,$$

we find

$$\begin{aligned}
&S_\sigma[(\widetilde{X}\,|\,\widetilde{\Gamma});\widetilde{\mathrm{A}},\gamma]-S_\sigma[(X\,|\,\Gamma);\mathrm{A},\gamma]\\
=\;&-\tfrac{1}{2}\int_\triangle\big[\mathrm{g}\big(\widetilde{X}(\cdot)\big)\big(D_{\widetilde{\mathrm{A}}}\widetilde{X}\overset{\wedge}{,}\star_\gamma D_{\widetilde{\mathrm{A}}}\widetilde{X}\big)(\cdot)-\mathrm{g}\big(\chi_1.\widetilde{X}(\cdot)\big)\big(D_{{}^{\chi_1}\widetilde{\mathrm{A}}}(\chi_1.\widetilde{X})\overset{\wedge}{,}\star_\gamma D_{{}^{\chi_1}\widetilde{\mathrm{A}}}(\chi_1.\widetilde{X}\big)(\cdot)\big]\\
&+\int_\triangle\big[\mathrm{B}\big(\widetilde{X}(\cdot)\big)-{}^M\ell^*\mathrm{B}(\chi_1,\widetilde{X})(\cdot)+\kappa_A\big(\widetilde{X}(\cdot)\big)\wedge\widetilde{\mathrm{A}}^A(\cdot)-{}^M\ell^*\kappa_A(\chi_1,\widetilde{X})(\cdot)\wedge{}^{\chi_1}\widetilde{\mathrm{A}}^A(\cdot)\big]\\
&-\tfrac{1}{2}\int_\triangle\big[\mathrm{c}_{AB}\big(\widetilde{X}(\cdot)\big)\big(\widetilde{\mathrm{A}}^A\wedge\widetilde{\mathrm{A}}^B\big)(\cdot)-{}^M\ell^*\mathrm{c}_{AB}(\chi_1,\widetilde{X})(\cdot)\,\big({}^{\chi_1}\widetilde{\mathrm{A}}^A\wedge{}^{\chi_1}\widetilde{\mathrm{A}}^B\big)(\cdot)\big]\\
&+\int_{\ell_3}(E_{\chi_2}-E_{\chi_1\cdot\chi_2})\big(\cdot,\widetilde{X}(\cdot)\big)+\int_{\ell_2}E_{\chi_1}\big(\cdot,\widetilde{X}(\cdot)\big)-\int_{\ell_1}E_{\chi_1}\big(\cdot,\widetilde{X}(\cdot)\big)\\
&+f_{\chi_1,\chi_2}\big(\widetilde{X}(v_2)\big)-f_{\chi_1,\chi_2}\big(\widetilde{X}(v_1)\big)\,,
\end{aligned}$$

with

$$f_{\chi_1,\chi_2}:=\big((\chi_1,\chi_2)\times\mathrm{id}_M\big)^*f\,,$$

the latter satisfying the defining relation

$$\mathsf{d}f_{\chi_1,\chi_2}(\sigma,m)=E_{\chi_1\cdot\chi_2}(\sigma,m)-E_{\chi_2}(\sigma,m)-\mathrm{e}^{-\overline{\chi_2^*\theta_L(\sigma)}}\cdot\big(\mathrm{id}_\Sigma\times{}^M\ell_{\chi_2(\sigma)}\big)^*E_{\chi_1}(\sigma,m)\,.\tag{8.35}$$

Reasoning as in the discussion of the DGC for the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect, we reduce the above expression to the form

$$\begin{aligned}
&S_\sigma[(\widetilde{X}\,|\,\widetilde{\Gamma});\widetilde{\mathrm{A}},\gamma]-S_\sigma[(X\,|\,\Gamma);\mathrm{A},\gamma]\\
=\;&\int_\triangle\big[\mathrm{B}\big(\widetilde{X}(\cdot)\big)-{}^M\ell^*\mathrm{B}(\chi_1,\widetilde{X})(\cdot)+\rho_{\chi_1^*\theta_L}\big(\cdot,\widetilde{X}(\cdot)\big)\big]+\int_{\ell_3}(E_{\chi_2}-E_{\chi_1\cdot\chi_2})\big(\cdot,\widetilde{X}(\cdot)\big)\\
&+\int_{\ell_2}E_{\chi_1}\big(\cdot,\widetilde{X}(\cdot)\big)-\int_{\ell_1}E_{\chi_1}\big(\cdot,\widetilde{X}(\cdot)\big)+f_{\chi_1,\chi_2}\big(\widetilde{X}(v_2)\big)-f_{\chi_1,\chi_2}\big(\widetilde{X}(v_1)\big)\\
=\;&\int_\triangle\mathsf{d}E_{\chi_1}\big(\cdot,\widetilde{X}(\cdot)\big)+\int_{\ell_3}(E_{\chi_2}-E_{\chi_1\cdot\chi_2})\big(\cdot,\widetilde{X}(\cdot)\big)+\int_{\ell_2}E_{\chi_1}\big(\cdot,\widetilde{X}(\cdot)\big)-\int_{\ell_1}E_{\chi_1}\big(\cdot,\widetilde{X}(\cdot)\big)\\
&+f_{\chi_1,\chi_2}\big(\widetilde{X}(v_2)\big)-f_{\chi_1,\chi_2}\big(\widetilde{X}(v_1)\big)\\
=\;&\int_{\ell_3}\big[(E_{\chi_2}-E_{\chi_1\cdot\chi_2})\big(\cdot,\widetilde{X}(\cdot)\big)+\big(\mathrm{id}_\Sigma\times{}^M\ell\big)^*E_{\chi_1}\big(\cdot,\chi_2(\cdot),\widetilde{X}(\cdot)\big)\big]+f_{\chi_1,\chi_2}\big(\widetilde{X}(v_2)\big)-f_{\chi_1,\chi_2}\big(\widetilde{X}(v_1)\big)\,.
\end{aligned}$$

Taking Eq. (8.35) into account, we finally arrive at the desired equality

$$S_\sigma[(\widetilde{X}\,|\,\widetilde{\Gamma});\widetilde{\mathrm{A}},\gamma] = S_\sigma[(X\,|\,\Gamma);\mathrm{A},\gamma]\,.$$

We conclude that the existence of γ ensures the existence of a *homomorphic* realisation of $C^\infty(\Sigma,\mathrm{G}_\sigma)$ on extended targets through equivalences. □

Prepared by the foregoing considerations, we now come to discuss the main point of the induction scheme. The latter is founded on the premise that defect junctions of valence greater than 3 can be fixed (up to an irremovable ambiguity quantified in the discussion surrounding Eqs. (2.77)-(2.79) in Ref. [RS09, Sec. 2.8]) in terms of the data of the elementary (*i.e.* trivalent) defect junctions in a limiting procedure applied to a defect graph obtained from the original one by an arbitrary resolution of its vertices of valence greater that or equal to 4 into trees of trivalent vertices. The resolution, first proposed in Ref. [RS09, Sec. 2.8] and elaborated in Ref. [Sus11, Rem. 5.6], is effected through introduction of intermediate defect lines whose length vanishes in the limit taken, *cf.* Figure 5. As discussed in the papers cited, internal consistency of the induction scheme, automatically inherited by defect junctions of valence greater than 4 from the 4-valent ones, is ensured by a cohomological (cocycle) constraint imposed upon the 2-isomorphism data carried by the trivalent defect junction. In the context of the gauged σ-model, we establish

Proposition 8.49. *Adopt the notation of Definition 3.2. The data carried by $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defects of Definition 8.43 and by the elementary $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect junctions of Definition 8.46 give rise to a simplicial string background in the sense of Ref.* [Sus11, Rem. 5.6], *with data of (component) $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump inter-bi-branes of valence greater than 3 induced, in the manner described ibidem, from those of the elementary $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump inter-bi-branes of the latter definition iff the G_σ-equivariant structure on the target $\mathcal{M}$ is coherent in the sense of Definition 8.10.*

Proof. Through a simple computation carried out along the lines of Ref. [RS09, Sec. 2.9], the validity of the induction scheme in the present setting is readily shown to be tantamount to the triviality of the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-valued **associator 3-cocycle**

$${}^M u^3_{\chi_1,\chi_2,\chi_3;\mathrm{A}} \coloneqq \gamma_{\chi_1,\chi_2\cdot\chi_3}\bullet[(\gamma_{\chi_2,\chi_3}\otimes\mathrm{Id})\circ\mathrm{Id}]\bullet\big(\mathrm{Id}\circ L^*_{\chi_3}\gamma_{\chi_1,\chi_2}\big)^{-1}\bullet\gamma^{-1}_{\chi_1\cdot\chi_2,\chi_3}$$

whose value at the image of the four-valent defect junction under the embedding map ξ measures the difference between the respective contributions of the two alternative sets of four-valent inter-bi-brane data to the action functional, obtained in the two limiting procedures shown in Figure 5.

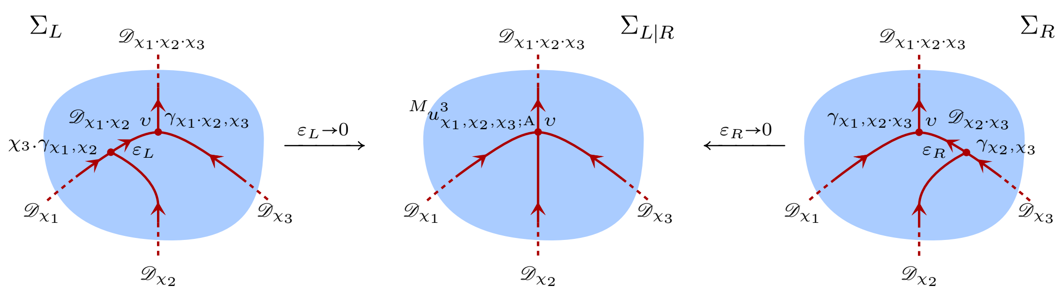

FIGURE 5. A four-valent defect vertex of a $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect network in $\Sigma_{L|R}$ obtained as a result of collapsing a pair of three-valent vertices in two inequivalent ways, whereby the two 2-morphisms φ^L and φ^R are induced at the vertex. These differ by the anomaly 3-cocycle (associator) ${}^M u^3_{\chi_1,\chi_2,\chi_2;\mathrm{A}}$.

The associator 3-cocycle is next identified with the pullback, along the map $(\chi_1,\chi_2,\chi_3)\times\mathrm{id}_M\ :\ \Sigma\times M\to\mathrm{G}^3_\sigma\times M$, of the anomaly 3-cocycle[22] ${}^M u^3\in Z^3\big(\pi_0(\mathrm{G}_\sigma),\mathrm{U}(1)^{\pi_0(M)}\big)$ of Ref. [GSW10, Cor. 6.7] and Ref. [GSW12, Cor. 11.7] whose class measures the obstruction to the existence of a *coherent* G_σ-equivariant structure on the bulk gerbe $\mathcal{G}$,

$${}^M u^3_{\chi_1,\chi_2,\chi_3;\mathrm{A}} = \big((\chi_1,\chi_2,\chi_3)\times\mathrm{id}_M\big)^*\,{}^M u^3\,.$$

[22]Strictly speaking, the associator 3-cocycle is the pullback of the preimage of the said $\mathrm{U}(1)^{\pi_0(M)}$-valued 3-cocycle on $\pi_0(\mathrm{G}_\sigma)$ with respect to the identification between the groups $H^0\big(\mathrm{G}^3_\sigma\times M,\mathrm{U}(1)\big)$ and $C^3\big(\pi_0(\mathrm{G}_\sigma),\mathrm{U}(1)^{\pi_0(M)}\big)$ discussed in the papers cited. We shall keep this identification implicit in what follows.

The vanishing of the said class is a sufficient and necessary condition for the coherence condition of Eq. (8.17) to be satisfied by the 2-isomorphism γ entering the definition of the trivalent $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect junction.

We conclude that the existence of a full-fledged G_σ-equivariant structure on the string background $\mathcal{M}$ of the mono-phase σ-model ensures the existence of an *associative* realisation of $C^\infty(\Sigma, \mathrm{G}_\sigma)$ on extended targets through equivalences, associated with topological world-sheet defects. □

Prior to taking up to the multi-phase case, we pause to emphasise the distinct status of the various elements of the construction of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network detailed above. Thus, while the existence of the topological $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect is ensured by the assumed $C^\infty(\Sigma, \mathrm{G}_\sigma)$-invariance of the gauged σ-model in the presence of a topologically trivial gauge field, the existence of the elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect junction and the validity of the induction scheme should be regarded as conditions necessary and sufficient for the existence of an extension of the construction of the gauge-symmetry defect to a consistent quantum field theory with the factorisation property, admitting a natural – from the physical point of view – induction scheme for multi-valent defect junctions. In view of the cohomological significance of the conditions, our findings mark the first step towards an explanation of the full-fledged large gauge anomaly in abstraction from the topological properties of the world-sheet gauge field, and in conformity with the infinitesimal symmetry structure of the σ-model captured by the small gauge anomaly.

Having completed the first stage of our construction for the gauged multi-phase σ-model, we may next pass to the investigation of conditions of coexistence of the conformal defect associated with the bi-brane of the original (gauged) σ-model and the newly introduced topological $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network. The analysis that follows splits into two steps: First of all, we set up a world-sheet description of a crossing between a $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect and a generic G_σ-transparent[23] domain wall that separates phases of the gauged σ-model. Secondly, we demand that the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network constructed previously remain topological in the presence of the domain wall. Accordingly, we begin with

Definition 8.50. Adopt the notation of Definitions I.2.6, 8.2 and 8.43, and of Proposition 8.7. Denote by

$$\mathcal{B}_{C^\infty(\Sigma,\mathrm{G}_\sigma);\mathrm{A}} := \mathcal{B}_{C^\infty(\Sigma,\mathrm{G}_\sigma)} \sqcup \mathcal{B}_\mathrm{A}$$

the composite bi-brane of the gauged multi-phase σ-model with an embedded $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network. Given an arbitrary map $\chi \in C^\infty(\Sigma, \mathrm{G}_\sigma)$ and a G_σ-transparent conformal defect $\mathscr{D}_\mathrm{A}$ carrying the data of the extended bi-brane $\mathcal{B}_\mathrm{A}$, the associated **elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump trans-defect junction $\mathcal{J}_{\chi;\mathrm{A}}$** for the gauged σ-model of Eq. (8.1) is the point $\jmath_{(4)} \subset \Sigma$ of intersection of a $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect $\mathscr{D}_\chi$ with $\mathscr{D}_\mathrm{A}$ of the type depicted in Figure 6, carrying the data of the **(component) elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump crossing inter-bi-brane** (the labelling of the defect lines converging at the junction starts with the bottom half of the vertical line in the figure, to which we assign label $(1,2)$, and continues – as usual – in the counter-clockwise direction)

$$\mathcal{J}_{\chi;\mathrm{A}} := \left(\{\chi\} \times \Sigma \times Q \equiv \Sigma \times Q; \pi_4^{1,2}, \pi_4^{2,3}, \pi_4^{3,4}, \pi_4^{4,1}; \Xi_\chi\right),$$

with inter-bi-brane maps

$$\begin{aligned}
\pi_4^{1,2} \quad &: \quad \{\chi\} \times \Sigma \times Q \to \Sigma \times Q \ : \ (\chi, \sigma, q) \mapsto (\sigma, \chi(\sigma).q)\,, \\
\pi_4^{2,3} \quad &: \quad \{\chi\} \times \Sigma \times Q \to \{\chi\} \times \Sigma \times M \ : \ (\chi, \sigma, q) \mapsto (\chi, \sigma, \iota_2(q))\,, \\
\pi_4^{3,4} \quad &: \quad \{\chi\} \times \Sigma \times Q \to \Sigma \times Q \ : \ (\chi, \sigma, q) \mapsto (\sigma, q)\,, \\
\pi_4^{4,1} \quad &: \quad \{\chi\} \times \Sigma \times Q \to \{\chi\} \times \Sigma \times M \ : \ (\chi, \sigma, q) \mapsto (\chi, \sigma, \iota_1(q))
\end{aligned}$$

and the 2-isomorphism

$$\Xi_\chi := (\chi \times \mathrm{id}_Q)^* \Xi \ : \ L_\chi^* \Phi_{\chi\mathrm{A}} \xRightarrow{\cong} \left(\underline{\iota}_2^* \Upsilon_\chi^{-1} \otimes \mathrm{Id}\right) \circ (\Phi_\mathrm{A} \otimes \mathrm{Id}) \circ \underline{\iota}_1^* \Upsilon_\chi\,,$$

where $\underline{\iota}_\alpha := \mathrm{id}_\Sigma \times \iota_\alpha$.

[23] By G_σ-transparency we mean preservation of the Noether charges of the global symmetry under gauging across the domain wall. As argued earlier, it is only in the presence of such distinguished defect lines that we can consistently gauge the symmetry G_σ.

Taking a disjoint union over the gauge group of component elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump crossing inter-bi-branes associated with various maps $\chi \in C^\infty(\Sigma, \mathrm{G}_\sigma)$, we obtain the **(total) elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump crossing inter-bi-brane**

$$\mathcal{J}_{C^\infty(\Sigma, \mathrm{G}_\sigma);\mathrm{A}} = \bigsqcup_{\chi \in C^\infty(\Sigma, \mathrm{G}_\sigma)} \mathcal{J}_{\chi;\mathrm{A}} \,.$$

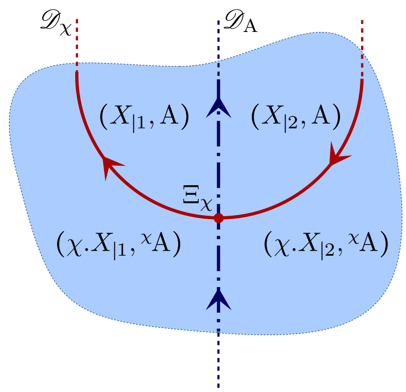

FIGURE 6. A four-valent crossing between a $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect $\mathscr{D}_\chi$ (red) and a generic G_σ-transparent defect of the gauged σ-model $\mathscr{D}_{\mathrm{A}}$ (dark blue). The crossing carries the data of the 2-isomorphism Ξ_χ.

Remark 8.51. Clearly, the consistency conditions of Eq. (I.2.1) for the inter-bi-brane maps are satisfied owing to the assumed G_σ-equivariance of the bi-brane maps.

As in the mono-phase setting, it is imperative for the interpretation of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect as a world-sheet realisation of the local symmetry of the gauged σ-model to ensure that the presence of the defect does not affect the action of the conformal group on field configurations. Consistently with the earlier discussion, this is amenable to direct verification which consists in determining a suitable (local) extension of the network-field configuration for the left-hand side of Figure 7 and checking that the value of the action functional does not change upon translating the defect junction along the defect $\mathscr{D}_{\mathrm{A}}$ to its new position as in the right-hand side of the same figure[24]. In this way, we establish

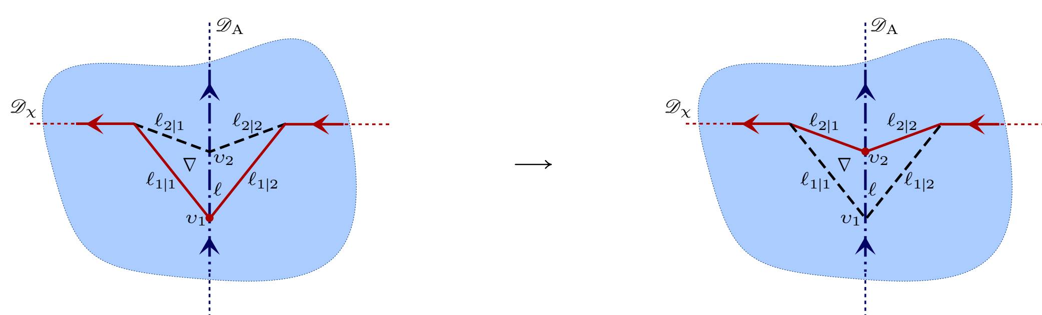

FIGURE 7. A homotopic displacement of a four-valent crossing between a $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect (red) and a generic G_σ-transparent defect of the gauged σ-model (dark blue) along the defect line of the latter.

[24]Since the defect $\mathscr{D}_{\mathrm{A}}$ is not, *a priori*, topological, we should only insist on invariance of the action functional under homotopic deformations of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect inducing translations of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump trans-defect junction along the defect line of $\mathscr{D}_{\mathrm{A}}$.

Proposition 8.52. *The network-field configuration for the defect $\mathscr{D}_\chi \sqcup \mathscr{D}_{\rm A}$ composed of the $C^\infty(\Sigma, {\rm G}_\sigma)$-jump defect $\mathscr{D}_\chi$ of Definition 8.43 and of an arbitrary ${\rm G}_\sigma$-transparent conformal defect $\mathscr{D}_{\rm A}$ is extendible in a neighbourhood of the elementary $C^\infty(\Sigma, {\rm G}_\sigma)$-jump trans-defect junction $\mathcal{J}_{\chi;{\rm A}}$ of Definition 8.50 in such a manner as to ensure that the $C^\infty(\Sigma, {\rm G}_\sigma)$-jump defect remains topological, in the sense of Ref.* [RS09, Sec. 2.9]*, also in the presence of $\mathscr{D}_{\rm A}$.*

Proof. For the sake of simplicity, we shall only consider homotopic deformations of the $C^\infty(\Sigma, {\rm G}_\sigma)$-jump defect of the sort depicted in Figure 7, leaving a verification of the claim in the case of a topological defect $\mathscr{D}_{\rm A}$ (in which also the latter could be deformed) to the reader. We shall also assume a cohomologically trivial ${\rm G}_\sigma$-equivariant string background, with a target as in Eqs. (8.33) and (8.34), and with a bi-brane

$$\mathcal{B} = (Q, \iota_1, \iota_2, \omega, \Phi)\,, \qquad\qquad \Phi := J_P$$

endowed with a ${\rm G}_\sigma$-equivariant structure

$$\Xi = \zeta \in C^\infty\left({\rm G}_\sigma \times Q, \mathbb{R}\right)\,,$$

so that we end up with a smooth function

$$\Xi_\chi = (\chi \times {\rm id}_Q)^* \zeta =: \zeta_\chi \in C^\infty\left(\Sigma \times Q, \mathbb{R}\right)$$

satisfying the defining equation

$$\mathsf{d}\zeta_\chi(\sigma, q) = P(\sigma, q) - \left({\rm id}_\Sigma \times {}^Q\ell\right)^* P(\sigma, \chi, q) + E_\chi\left(\sigma, \iota_1(q)\right) - E_\chi\left(\sigma, \iota_2(q)\right) + \lambda_{\chi^*\theta_L}(\sigma, q)\,.$$

Define an extension of the network-field configuration $(X\,|\,\Gamma)$ for the left-hand side of Figure 7 by the following formulæ:

$$\widehat{\xi}_{\ell_1} := \left({\rm id}_\Sigma, \widehat{X}_{\ell_1}\right) \quad : \quad \nabla \smallsetminus \ell \to \{\chi\} \times (\nabla \smallsetminus \ell) \times M \;:\; \sigma \mapsto (\chi, \sigma, X(\sigma))\,,$$

$$\widehat{\xi}_{v_1} := \left({\rm id}_\Sigma, \widehat{X}_{v_1}\right) \quad : \quad \ell \to \{\chi\} \times \ell \times Q \;:\; \sigma \mapsto (\chi, \sigma, X(\sigma))\,,$$

and subsequently use it to write a network-field configuration $(\widetilde{X}\,|\,\widetilde{\Gamma})$ for the right-hand side of the same figure as

$$\widetilde{X}|_{\Sigma \smallsetminus \nabla} = X|_{\Sigma \smallsetminus \nabla}\,, \qquad \widetilde{X}|_{\nabla \smallsetminus (\ell_{2|1} \cup \ell_{2|2} \cup \ell)} = \chi.\widehat{X}_{\ell_1}|_{\nabla \smallsetminus (\ell_{2|1} \cup \ell_{2|2} \cup \ell)}\,,$$

$$\widetilde{X}|_{(\ell_{2|1} \cup \ell_{2|2}) \smallsetminus \{v_2\}} = \widehat{X}_{\ell_1}|_{(\ell_{2|1} \cup \ell_{2|2}) \smallsetminus \{v_2\}}\,, \qquad \widetilde{X}|_{\ell \smallsetminus \{v_2\}} = \chi.\widehat{X}_{v_1}|_{\ell \smallsetminus \{v_2\}}\,, \qquad \widetilde{X}|_{\{v_2\}} = \widehat{X}_{v_1}|_{\{v_2\}}\,.$$

This is to be augmented by the definition of the new gauge field,

$$\widetilde{\rm A}|_{\Sigma \smallsetminus \nabla} = {\rm A}|_{\Sigma \smallsetminus \nabla}\,, \qquad \widetilde{\rm A}|_{\nabla \smallsetminus \partial\nabla} = {}^\chi{\rm A}|_{\nabla \smallsetminus \partial\nabla}\,.$$

Repeating previous arguments, we readily establish

$$\begin{aligned}
&S_\sigma[(\widetilde{X}\,|\,\widetilde{\Gamma}); \widetilde{\rm A}, \gamma] - S_\sigma[(X\,|\,\Gamma); {\rm A}, \gamma] \\
=\;& \int_\nabla \left[{}^M\ell^*{\rm B}(\chi, X)(\cdot) - {\rm B}\left(X(\cdot)\right) + {}^M\ell^*\kappa_A(\chi, X)(\cdot) \wedge {}^\chi{\rm A}^A(\cdot) - \kappa_A\left(X(\cdot)\right) \wedge {\rm A}^A(\cdot)\right] \\
& -\tfrac{1}{2} \int_\nabla \left[{}^M\ell^*{\rm c}_{AB}(\chi, X)(\cdot)\left({}^\chi{\rm A}^A \wedge {}^\chi{\rm A}^B\right)(\cdot) - {\rm c}_{AB}\left(X(\cdot)\right)\left({\rm A}^A \wedge {\rm A}^B\right)(\cdot)\right] \\
& + \int_\ell \left[\left({\rm id}_\Sigma \times {}^Q\ell\right)^* P\left(\cdot, \chi(\cdot), X(\cdot)\right) - P\left(\cdot, X(\cdot)\right) - {}^Q\ell^* k_A(\chi, X)(\cdot)\,{}^\chi{\rm A}^A(\cdot) + k_A\left(X(\cdot)\right){\rm A}^A(\cdot)\right] \\
& + \int_{\ell_{2|1} \cup \ell_{2|2} \cup (-\ell_{1|1}) \cup (-\ell_{1|2})} E_\chi\left(\cdot, X(\cdot)\right) + \zeta_\chi\left(v_2, X(v_2)\right) - \zeta_\chi\left(v_1, X(v_1)\right) \\
=\;& -\int_{\ell_{2|1} \cup \ell_{2|2} \cup (-\ell_{1|1}) \cup (-\ell_{1|2})} E_\chi\left(\cdot, X(\cdot)\right) - \int_\ell \left[E_\chi\left(\cdot, \iota_1 \circ X(\cdot)\right) - E_\chi\left(\cdot, \iota_2 \circ X(\cdot)\right)\right] \\
& + \int_\ell \left[\left({\rm id}_\Sigma \times {}^Q\ell\right)^* P\left(\cdot, \chi(\cdot), X(\cdot)\right) - P\left(\cdot, X(\cdot)\right) - \lambda_{\chi^*\theta_L}\left(\cdot, X(\cdot)\right)\right] \\
& + \int_{\ell_{2|1} \cup \ell_{2|2} \cup (-\ell_{1|1}) \cup (-\ell_{1|2})} E_\chi\left(\cdot, X(\cdot)\right) + \zeta_\chi\left(v_2, X(v_2)\right) - \zeta_\chi\left(v_1, X(v_1)\right) \\
=\;& -\int_\ell \mathsf{d}\zeta_\chi\left(\cdot, X(\cdot)\right) + \zeta_\chi\left(v_2, X(v_2)\right) - \zeta_\chi\left(v_1, X(v_1)\right) = 0\,,
\end{aligned}$$

which is the desired result. $\square$

So far, no structure beyond that which is required for the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-invariance of the gauged σ-model in the presence of G_σ-transparent defects was necessary. However, in order to ensure topologicality of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network, we should also demand invariance of the action functional under homotopies of the network that pull the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect junction across $\mathscr{D}_{\mathrm{A}}$ as in Figure 8. This imposes familiar constraints upon the data carried by the two crossing defect

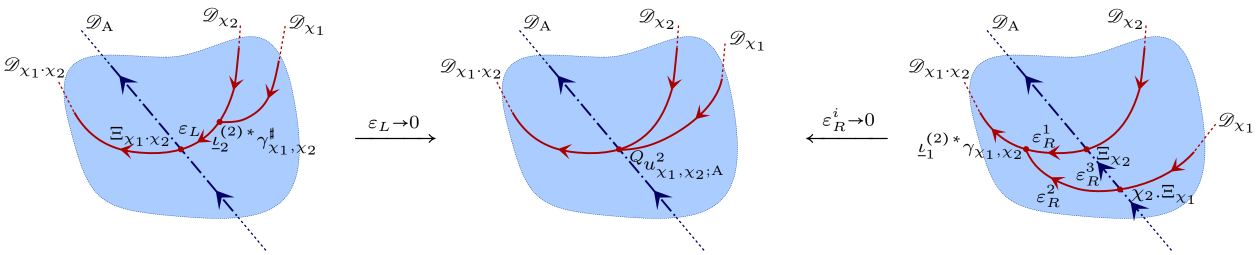

FIGURE 8. Pulling a tri-valent defect vertex of a $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network across a generic G_σ-transparent defect line yields the anomaly 3-cocycle ${}^Q u^2_{\chi_1,\chi_2;\mathrm{A}}$.

networks.

Proposition 8.53. *Adopt the notation of Definition 3.2 and assume that the conditions stated in Proposition 8.49 are satisfied. The network-field configuration for an arbitrary graph of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defects of Definition 8.43 crossing a G_σ-transparent conformal defect $\mathscr{D}_{\mathrm{A}}$ is extendible (as long as there are no topological obstructions within the world-sheet) in a neighbourhood of the crossing in such a manner as to ensure that the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network remains topological, in the sense of Ref.* [RS09, Sec. 2.9]*, also in the presence of $\mathscr{D}_{\mathrm{A}}$ iff the G_σ-equivariant structure on the bi-brane $\mathcal{B}$ is coherent in the sense of Definition 8.10.*

Proof. Reasoning along the same lines as in the case of Figure 5, we find out that the elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump trans-defect junction can be pulled through $\mathscr{D}_{\mathrm{A}}$ iff the **cross-multiplication 2-cocycle**

$${}^Q u^2_{\chi_1,\chi_2;\mathrm{A}} := (\mathrm{Id}\circ\underline{\iota}_1^*\gamma_{\chi_1,\chi_2})\bullet(\mathrm{Id}\circ(\Xi_{\chi_2}\otimes\mathrm{Id})\circ\mathrm{Id})\bullet L^*_{\chi_2}\Xi_{\chi_1}\bullet\Xi^{-1}_{\chi_1\cdot\chi_2}\bullet\big(\big(\underline{\iota}_1^*\gamma^\sharp_{\chi_1,\chi_2}\otimes\mathrm{Id}\big)\circ\mathrm{Id}\big)^{-1}$$

trivialises in cohomology.

The latter is the pullback, along the map $(\chi_1,\chi_2)\times\mathrm{id}_Q\,:\,\Sigma\times Q\to\mathrm{G}_\sigma^2\times Q$, of the anomaly 2-cocycle ${}^Q u^2\in Z^2\big(\pi_0(\mathrm{G}_\sigma),\mathrm{U}(1)^{\pi_0(Q)}\big)$ of Ref. [GSW12, Cor. 11.21] whose class measures the obstruction to the existence of a *coherent* G_σ-equivariant structure on the bi-brane $\mathcal{B}$,

$${}^Q u^2_{\chi_1,\chi_2;\mathrm{A}} = ((\chi_1,\chi_2)\times\mathrm{id}_Q)^*\,{}^Q u^2\,.$$

Triviality of ${}^Q u^2_{\chi_1,\chi_2;\mathrm{A}}$ is a sufficient and necessary condition for the coherence condition of Eq. (8.18) to be satisfied by the 2-isomorphism Ξ entering the definition of the elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump trans-defect junction.

We conclude that the existence of a full-fledged G_σ-equivariant structure on the string background of a multi-phase σ-model on a world-sheet Σ with circular (*i.e.* non-intersecting) G_σ-transparent defect lines ensures that the associative realisation of $C^\infty(\Sigma, \mathrm{G}_\sigma)$ mentioned in the proof of Proposition 8.49 extends to the multi-phase setting. □

In the last stage of the construction, it remains to examine the fate of the topologicality of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect network in the presence of arbitrary defect junctions of G_σ-transparent defects of the gauged multi-phase σ-model. This boils down to calculating the correction to the action functional induced in the process of pulling the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect past a vertex of the G_σ-transparent defect network as, *e.g.*, in Figure 9. As expected, we obtain

Proposition 8.54. *Adopt the notation of Definition 3.2 and assume that the conditions stated in Proposition 8.53 are satisfied. The network-field configuration for an arbitrary graph of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defects of Definition 8.43 crossing a network of G_σ-transparent conformal defects is extendible (as long as there are no topological obstructions within the world-sheet) in a neighbourhood of every vertex*

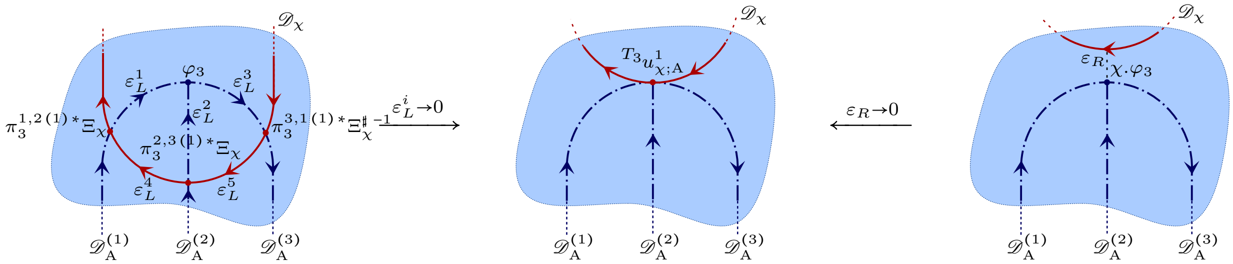

FIGURE 9. Pulling a $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect past a three-valent defect junction of a generic G_σ-transparent defect network yields the anomaly 3-cocycle ${}^{T_3}u^1_{\chi;\mathrm{A}}$.

of the latter network in such a manner as to ensure that the $C^\infty(\Sigma,\mathrm{G}_\sigma)$-jump defect network remains topological, in the sense of Ref. [RS09, Sec. 2.9]*, also in the presence of junctions of the* G_σ*-transparent defects.*

Proof. The claim of the proposition follows directly from cohomological triviality of the **intertwiner 1-cocycle** expressible as the pullback, along the map $\chi\times\mathrm{id}_{T_n}\ :\ \Sigma\times T_N\to\mathrm{G}_\sigma\times T_N$, of the anomaly 1-cocycle ${}^{T_n}u^2\in Z^2\left(\pi_0(\mathrm{G}_\sigma),\mathrm{U}(1)^{\pi_0(T_n)}\right)$ of Ref. [GSW12, Cor. 11.26],

$$ {}^{T_n}u^1_{\chi;\mathrm{A}}=(\chi\times\mathrm{id}_Q)^*\,{}^{T_n}u^1\,. $$

The class of the latter measures the obstruction to the existence of a *coherent* G_σ-equivariant structure on the complete string background, as expressed by Eq. (8.13).

We conclude that the existence of a full-fledged G_σ-equivariant structure on the string background of a multi-phase σ-model on a world-sheet Σ with arbitrary G_σ-transparent defect lines ensures that the associative realisation of $C^\infty(\Sigma,\mathrm{G}_\sigma)$ mentioned in the proof of Proposition 8.49 extends to the multi-phase setting. □

Remark 8.55. We close the present section with a comment on how the hitherto findings can be employed to reconstruct the gauged σ-model coupled to an arbitrary gauge bundle $\mathcal{P}_{\mathrm{G}_\sigma}$ with connection over Σ out of data of the trivial gauge bundle and those of a topological $C^\infty(\mathsf{N}^1\mathcal{O}_\Sigma,\mathrm{G}_\sigma)$-jump defect network, defined over the nerve $\mathsf{N}^\bullet\mathcal{O}_\Sigma$ of an open cover $\mathcal{O}_\Sigma$ of the world-sheet Σ. Rather than formalising our discussion, we illustrate the general idea by referring to the generic local world-sheet situation depicted in Figure 10. The latter shows a piece of the world-sheet covered by a number of elements of an open cover $\mathcal{O}_\Sigma=\{\Sigma_i\}_{i\in\mathscr{I}}$ over which the principal G_σ-bundle is assumed to trivialise, so that, *e.g.*, $\mathsf{P}_{\mathrm{G}_\sigma}|_{\Sigma_i}\cong\Sigma_i\times\mathrm{G}_\sigma$ and the principal G_σ-connection 1-form induces a locally smooth 1-form on the base $\mathrm{A}_i\in\Omega^1(\Sigma_i)\otimes\mathfrak{g}_\sigma$.

We set up the local description as follows: Assume given an embedded defect quiver $\Gamma\subset\Sigma$. Consider an oriented trivalent graph $\Gamma_{\mathcal{O}_\Sigma}\subset\Sigma$ that is **Γ-transversal and Γ-simple, and subordinate to $\mathcal{O}_\Sigma$** in the sense that $\Gamma\cap\Gamma_{\mathcal{O}_\Sigma}$ is discrete (*i.e.* composed of a finite number of intersection points) and does not contain vertices of Γ, and is such that for every edge of the graph there exists a pair $(i,j)\in\mathscr{I}^2$ of indices of the cover with the property that the edge is contained in $\Sigma_{ij}=\Sigma_i\cap\Sigma_j$. Clearly, to every vertex of the graph, we may associate a triple $(i,j,k)\in\mathscr{I}^3$ of indices corresponding to the three edges converging at the vertex. The graph splits Σ into a collection of disjoint plaquettes. Each of them will be labelled by the index of the element of the open cover in which it is contained, *e.g.*, $p_i\subset\Sigma_i$. We label the edge separating plaquettes p_i and p_j with the two indices i and j written in the order determined by the orientation of the edge as in the case of the edge e_{ji} in the figure. Furthermore, we label each vertex of the graph by an arbitrary cyclic permutation of the three indices associated to the edges meeting at the vertex, read off anti-clockwise around the vertex, as in the case of the vertex v_{lmo} in the figure. Finally, each junction and each (segment of a) defect line of Γ is labelled by the index of the plaquette in which it sits, and every (4-valent) crossing between an edge of $\Gamma_{\mathcal{O}_\Sigma}$ and an edge of Γ by the pair of indices assigned to the edge of $\Gamma_{\mathcal{O}_\Sigma}$ going through it, as the junction $\jmath_n$, the segment s_n and the crossing c_{on} in the figure, respectively.

Once the assignment of labels has been accomplished, and given a collection of local trivialisations $\tau_i:\pi^{-1}_{\mathcal{P}_{\mathrm{G}_\sigma}}(\Sigma_i)\to\Sigma_i\times\mathrm{G}_\sigma$ of $\mathcal{P}_{\mathrm{G}_\sigma}$, the attendant local connection 1-forms A_i and transition maps

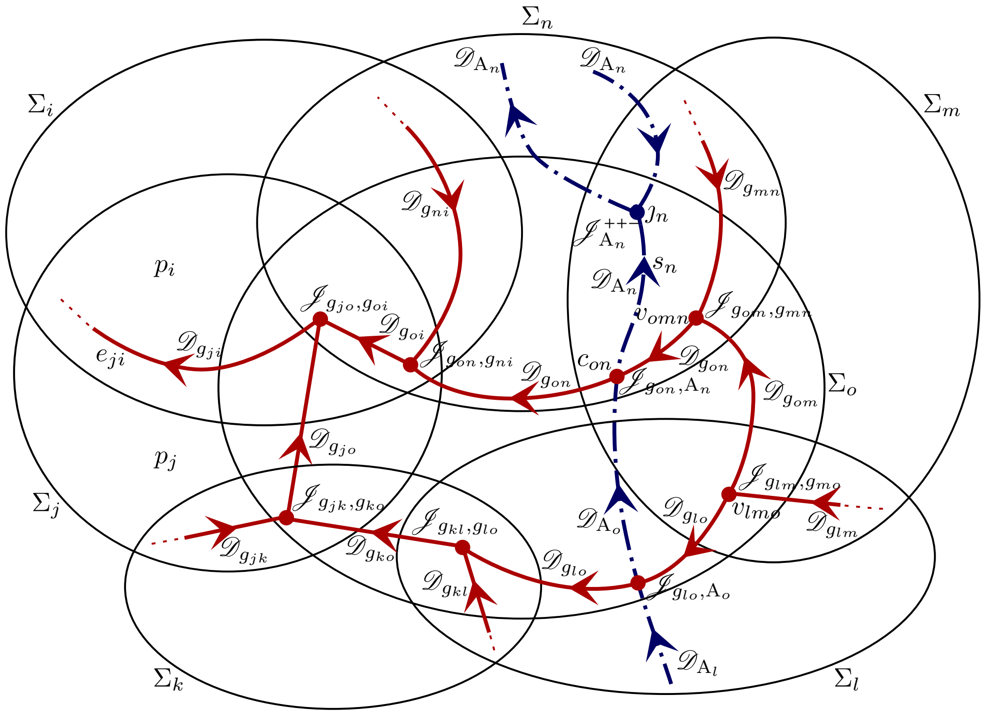

FIGURE 10. A reconstruction of an arbitrary principal G_σ-bundle with connection over the world-sheet through local trivialisation. The local description uses data of a topological $C^\infty(\mathsf{N}^1\mathcal{O}_\Sigma, \mathrm{G}_\sigma)$-jump defect network.

$g_{ij} : \Sigma_{ij} \to \mathrm{G}_\sigma$, and a global section $[(g_{ij}, X_i)]$ of the associated bundle $\mathsf{P}_{\mathrm{G}_\sigma} \times_{\mathrm{G}_\sigma} \mathscr{F}$, it is straightforward to associate (local) geometric objects to elements of the decorated triangulation of the world-sheet defined (as above) by $\Gamma_{\mathcal{O}_\Sigma}$. Thus, to a plaquette p_i we pull back, along the local section (id_{p_i}, X_i), the data of the extended background $\mathfrak{B}_{\mathrm{A}_i}$ (including the extended metric $\mathrm{g}_{\mathrm{A}_i}$, the extended gerbe $\mathcal{G}_{\mathrm{A}_i}$ *etc.*). In particular, we pull back the 1-isomorphism Φ_{A_n} to the segment s_n, and the 2-isomorphism $\varphi^{++-}_{\mathrm{A}_n}$ to the junction $\jmath_n$. Based on our previous considerations, we endow the edge e_{ji} with the structure of a **local transition defect** $\mathscr{D}_{g_{ji}}$ (denoted, by a slight abuse of the notation, by the same symbol as and) differing from the component $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump defect $\mathscr{D}_\chi$ of Definition 8.43 *exclusively* in the choice of the associated G_σ-valued map (g_{ji} *versus* χ), required to be smooth only *locally* in the present case. Similarly, we put over the crossing c_{on} the **local trans-defect transition junction** $\mathscr{J}_{g_{on},\mathrm{A}_n}$, which is none other than the elementary $C^\infty(\Sigma, \mathrm{G}_\sigma)$-jump trans-defect junction $\mathscr{J}_{\chi,\mathrm{A}}$ of Definition 8.50 with locally smooth data (g_{on}, A_n) instead of the globally smooth ones (χ, A). Up to now, we have been using only the structure necessitated by the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-invariant gauged σ-model coupled to the topologically trivial gauge field. The existence of the latter also ensures that local transition defects can be deformed homotopically (*i.e.* drawn arbitrarily) within the domains of definition of the respective transition maps without changing the value of the action functional, also along defect lines of Γ and past its junctions, so that, *e.g.*, we may pull e_{on} up past $\jmath_n$. This renders the field theory defined in terms of the local data introduced above independent of some of the arbitrary choices made in the trivialisation procedure. It is through the imposition of requirements of internal consistency of the ensuing field theory with local transition defects, and arguments of independence of the latter theory of the arbitrary choices made that we shall next rediscover the remaining components of the G_σ-equivariant structure on $\mathfrak{B}$ as necessary ingredients of the local construction of a topologically non-trivial gauge bundle over Σ coupled to $\mathfrak{B}$.

Given the assignment of geometric data to edges of $\Gamma_{\mathcal{O}_\Sigma}$, detailed arguments of Ref. [RS09, Sec. 2.7] force us to pull back to its vertices data of appropriate trivialising gerbe 2-isomorphisms, *cf.* Definition

I.2.1. Taking into account the cocycle condition satisfied by the transition maps of $\mathcal{P}_{\mathrm{G}_\sigma}$, we are thus led to require the existence of the 2-isomorphism $\gamma_{g_{lm},g_{mo}}$ for v_{lmo} (determined by the transition maps in the very same manner as the 2-isomorphisms γ_{χ_1,χ_2} of Definition 8.46 are determined by the gauge maps χ_1, χ_2), and so – ultimately, in view of the arbitrariness of the transition functions – the existence of the underlying 2-isomorphism γ of Definition 8.10. The **local transition defect junctions** $\mathscr{J}_{g_{lm},g_{mo}}$ thus obtained can be moved around within the domain Σ_{lmo} of their definition at no cost in the value of the action functional as long as they do not cross a defect line of Γ, and so it remains to ensure that this feature prevails also in the presence of the defect lines (so that we may, *e.g.*, pull v_{omn} across s_n in the figure), and that the field theory determined by the local data of the trivialisation of $\mathcal{P}_{\mathrm{G}_\sigma}$ including the local transition defect junctions does not suffer from any ambiguities under refinement of a given open cover or a simple change of the choice of indices in quadruple and higher-order intersections of elements of $\mathcal{O}_\Sigma$ (as, *e.g.*, in Σ_{ijon} in the figure). As the discussion conveyed in the context of the implementation of the $C^\infty(\Sigma, \mathrm{G}_\sigma)$-action through defects indicates, we need the coherence condition (8.18) for the former, and the standard argument for the quadruple intersection (used previously in the context of the associativity of the world-sheet realisation of $C^\infty(\Sigma, \mathrm{G}_\sigma)$) demonstrates the necessity (and sufficiency) of imposing the coherence condition (8.17). It is now clear that the systematic procedure leads to a reconstruction of a consistent coupling of the non-trivial principal G_σ-bundle $\mathcal{P}_{\mathrm{G}_\sigma}$ over the world-sheet to the original string background, and yields a gauged σ-model manifestly independent of the arbitrary choices entering its local description. Our analysis shows, once again, that the passage from trivial to non-trivial gauge bundles coupled to the string background $\mathfrak{B}$ of the σ-model with a global G_σ-symmetry (with a vanishing small gauge anomaly) does *necessitate* the existence of a full-fledged G_σ-equivariant structure on $\mathfrak{B}$.

The findings of the last section (and those of the previous one) are summarised in

Theorem 8.56. *The gauged non-linear two-dimensional σ-model coupled to gauge fields of an arbitrary topology, whose incorporation is necessary to account for the existence of G_σ-twisted (network-)field configurations in the non-linear two-dimensional σ-model with the target space given by the orbit space of the parent σ-model with respect to the action of a group G_σ of rigid symmetries of the latter σ-model, exists iff the string background of the parent σ-model can be endowed with a G_σ-equivariant structure.*

9. Conclusions and outlook

The paper gives an account of a comprehensive treatment of algebraic and differential-geometric aspects of rigid symmetries of the multi-phase two-dimensional non-linear σ-model and of their gauging, laying due emphasis on the underlying gerbe theory and – also in this latter context – exploiting the interplay between σ-model dualities and conformal defects. It develops a scheme of description of the said symmetries based on the concept of the (relative) generalised geometry and thus naturally adapted to the setting of the target space of the σ-model endowed with the structure of the 2-category of bundle gerbes with connection over it, discusses the transgression of that scheme to the phase space of the σ-model, and – finally – extracts from it a simple geometric measure of the gauge anomaly that obstructs an attempt at rendering the rigid symmetries local. The naturalness of this measure is subsequently corroborated in the framework of the theory of principal bundles with a structural action groupoid over the world-sheet of the σ-model, leading to a systematic construction of topological defect networks implementing the action of the gauge group as well as those realising a local (world-sheet) trivialisation of a gauge bundle of an arbitrary topology in the gauged multi-phase σ-model. The latter construction demonstrates the necessity of the existence of a full-fledged equivariant structure on the string background of the σ-model for a consistent gauging of its rigid symmetries.

For the sake of concreteness, and by way of a concise summary, we list the main results of our work hereunder.

(1) The definition of an algebroidal target-space model of the Poisson algebra of Noether charges of a rigid symmetry, inspired by earlier work of Alekseev and Strobl, and that of Hitchin and Gualtieri, and formulated in terms of a twisted bracket structure on the space of sections of generalised tangent bundles over the target space. It is based on the following:
   - a reconstruction of the model through the study of infinitesimal lagrangean symmetries (Propositions 2.19, 5.5 and 6.1; Corollary 3.4);
   - a classification of its automorphisms (Propositions 2.5 and 5.2);

- a reformulation in terms of local data of the 2-category of abelian bundle gerbes with connection over the target space of the σ-model, *via* Hitchin-type isomorphisms (Corollary 2.17; Proposition 5.3), compatible with the action of gerbe morphisms (Theorem 4.1);
- a homomorphic transgression to the state space of the σ-model (Theorems 3.11 and 5.8; Propositions 3.12 and 5.9), consistent with the defect-duality correspondence (Proposition 4.2).

(2) A description of the relative (co)homology of the hierarchical target space of the σ-model and a simple reinterpretation of the physically motivated bracket structure from the previous point. These include

- an elementary characterisation of the relative singular (co)homology (Proposition 7.10), alongside its realisation in terms of differential forms (Theorem 7.14);
- introduction of a relative variant of the Cartan calculus and identification of the associated (relative) twisted Courant algebroid as the aforementioned bracket structure (Proposition 7.17; Theorem 7.19).

(3) A canonical description of rigid and gauged symmetries of the σ-model in the first-order formalism of Ref. [Sus11]. Here, we present

- an investigation of conditions of continuity of Noether charges across conformal defects and of their additive conservation in trans-defect (resp. twisted-sector) splitting-joining interactions (Propositions 4.5 and 4.6; Theorems 6.3 and 6.5);
- a reinterpretation of the small gauge anomaly as an obstruction to the existence of a hamiltonian realisation of the symmetry algebra on states of the σ-model, consistent with interactions (Theorem 8.17), resp. to a canonical realisation of the (infinitesimal) gauge symmetry on the state space of the gauged σ-model through elements of the characteristic distribution of the relevant presymplectic form (Theorem 8.18).

(4) An investigation of the Lie-groupoidal geometry of the gauge anomaly. It yields

- a reinterpretation of the small gauge anomaly in the algebroidal framework introduced (as a combined Leibniz, Jacobi and involutivity anomaly obstructing the existence of a Lie algebroid within the relative twisted Courant algebroid), from which there emerges the tangent algebroid of the action groupoid associated with the action of the symmetry group G_σ on the target space $\mathscr{F}$ of the σ-model (Theorems 8.21 and 8.25);
- an elucidation of the latter phenomenon through a categorial equivalence between – on the one hand – a category formed from fundamental structures of a consistent gauged σ-model (a principal G_σ-bundle over the world-sheet with the property that the bundle associated to it through a G_σ-action on $\mathscr{F}$ admits a global section) and morphisms between them and – on the other hand – the groupoid of principal bundles over the world-sheet with the structural action groupoid $\mathrm{G}_\sigma \ltimes \mathscr{F}$ whose appearance is the first hint of a local world-sheet description of an orbit space of $\mathscr{F}$ with respect to the action of G_σ (Theorem 8.41);
- an extension of the said equivalence to the setting with connection through an explicit construction of a topological defect network implementing gauge transformations (defined globally or only locally) on states of the gauged σ-model, giving rise to a hands-on realisation of the concept of a simplicial duality background of Ref. [Sus11, Rem. 5.6] (Section 8.3);
- a reinterpretation of the large gauge anomaly as an obstruction to the existence of a consistent quantum CFT of the gauged σ-model with topological gauge-symmetry defects resp. to the existence of the gauged σ-model coupled to a gauge field of an arbitrary topology whose indispensable incorporation in a unified field-theoretic framework is understood from a purely geometric point of view (Proposition 8.54; Remark 8.55; Theorem 8.56).

The study reported in the present paper, taken in conjunction with the earlier works on the subject, and in particular Refs. [GSW11, RS09, Sus11, GSW10, GSW12] (*cf.* also the references listed there), of which it constitutes a natural completion, leaves us with a fairly good understanding of the deeper nature of rigid symmetries of the multi-phase σ-model. It also motivates and lays the groundwork for a number of new lines of research, of which we mention the following:

(1) A systematic construction of *all* bi-branes of the coset (resp. gauged) σ-model (upon relaxing, in particular, the restrictive assumption of abelianness of the associated gerbe bimodules, *cf.* Ref. [Gaw05]), and comparison of its results with predictions of the categorial quantisation scheme[25].

[25]The author is grateful to Ingo Runkel for raising this point in a private discussion.

(2) Application of the gauge principle in the study of T-duality in the context of the gerbe theory of the σ-model.
(3) A world-sheet construction of the σ-model on an orbit space of the target space with respect to the action of *bona fide* dualities, based on the defect-duality correspondence (*e.g.*, T-folds).
(4) Incorporation of world-sheet/target-space supersymmetry into the gerbe-theoretic framework of description of the σ-model, with direct reference to the concept of a pure spinor but also with view to deriving a 2-categorially-twisted (relative) extension of Gualtieri's generalised complex geometry of its target space.
(5) Study of relations between gauged multi-phase σ-models and Poisson σ-models in the context of the underlying algebroidal structure over the target space (drawing inspiration from but also going beyond the long-known correspondence between gauged WZW models and certain distinguished Poisson σ-models).

We hope to return to these problems in near future.

References

[AS05] A.Yu. Alekseev and T. Strobl, *"Current algebra and differential geometry"*, JHEP **0503** (2005), 035.

[Bar07] D. Baraglia, *"Generalized geometry"*, Ph.D. thesis, The University of Adelaide, Australia, 2007.

[BT82] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics, vol. 82, Springer, 1982.

[Cha98] D.S. Chatterjee, *"On gerbs"*, Ph.D. thesis, Trinity College, Cambridge University, UK, 1998.

[Cou90] T.J. Courant, *"Dirac manifolds"*, Trans. Amer. Math. Soc. **319** (1990), 631–661.

[Dor87] I.Ya. Dorfman, *"Dirac structures of integrable evolution equations"*, Phys. Lett. **A125** (1987), 240–246.

[Dor93] ———, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, Nonlinear science: Theory and Applications Series, John Wiley & Sons, 1993.

[FFRS04] J. Fröhlich, J.A. Fuchs, I. Runkel, and C. Schweigert, *"Kramers–Wannier duality from conformal defects"*, Phys. Rev. Lett. **93** (2004), 070601.

[FFRS07] ———, *"Duality and defects in rational conformal field theory"*, Nucl. Phys. **B763** (2007), 354–430.

[FFRS09] ———, *"Defect lines, dualities, and generalised orbifolds"*, Proceedings of the XVIth Iinternational Congress on Mathematical Physics: Prague, Czech Republic, 3-8 August 2009 (P. Exner, ed.), World Scientific, 2009, pp. 608–613.

[FG94] J. Fröhlich and K. Gawędzki, *"Conformal Field Theory and Geometry of Strings"*, Vancouver 1993, Proceedings, Mathematical Quantum Theory I: Field Theory and Many-Body Theory (J. Feldman, R. Froese, and L.M. Rosen, eds.), CRM Proceedings & Lecture Notes, vol. 7, American Mathematical Society, 1994, pp. 57–97.

[FNSW09] J.A. Fuchs, T. Nikolaus, C. Schweigert, and K. Waldorf, *"Bundle gerbes and surface holonomy"*, arXiv preprint: 0901.2085 [math.DG].

[FOS] J.M. Figueroa-O'Farrill and S. Stanciu, *"Equivariant cohomology and gauged bosonic σ-models"*, arXiv preprint: hep-th/9407149.

[FOS94] ———, *"Gauged Wess–Zumino terms and equivariant cohomology"*, Phys. Lett. **B341** (1994), 153–159.

[FOS01] ———, *"D-brane charge, flux quantisation and relative (co)homology"*, JHEP **0101** (2001), 006.

[FSS96] J.A. Fuchs, B. Schellekens, and C. Schweigert, *"The resolution of field identification fixed points in diagonal coset theories"*, Nucl. Phys. **B461** (1996), 371–406.

[FSW08] J.A. Fuchs, C. Schweigert, and K. Waldorf, *"Bi-branes: Target space geometry for world sheet topological defects"*, J. Geom. Phys. **58** (2008), 576–598.

[Gaw72] K. Gawędzki, *"On the geometrization of the canonical formalism in the classical field theory"*, Rep. Math. Phys. **3** (1972), 307–326.

[Gaw99] ———, *"Conformal field theory: A case study"*, arXiv preprint: hep-th/9904145.

[Gaw05] ———, *"Abelian and non-Abelian branes in WZW models and gerbes"*, Commun. Math. Phys. **258** (2005), 23–73.

[GR03] K. Gawędzki and N. Reis, *"Basic gerbe over non simply connected compact groups"*, J. Geom. Phys. **50** (2003), 28–55.

[GSW08] K. Gawędzki, R.R. Suszek, and K. Waldorf, *"WZW orientifolds and finite group cohomology"*, Commun. Math. Phys. **284** (2008), 1–49.

[GSW10] ———, *"Global gauge anomalies in two-dimensional bosonic sigma models"*, Comm. Math. Phys. **302** (2010), 513–580.

[GSW11] ———, *"Bundle gerbes for orientifold sigma models"*, Adv. Theor. Math. Phys. **15** (2011), 621–688.

[GSW12] ———, *"The gauging of two-dimensional bosonic sigma models on world-sheets with defects"*, arXiv preprint: 1202.5808 [hep-th], submitted to Rev. Math. Phys.

[Gua03] M. Gualtieri, *"Generalized Complex Geometry"*, Ph.D. thesis, St John's College, Oxford, UK, 2003.

[Hae84] A. Haefliger, *"Groupoïdes d'holonomie et classifiants"*, Astérisque **116** (1984), 70–97.

[Hit03] N. Hitchin, *"Generalized Calabi-Yau manifolds"*, Quart. J. Math. Oxford Ser. **54** (2003), 281–308.

[Hit06] ———, *"Brackets, forms and invariant functionals"*, Asian J. Math. **10** (2006), 541–560.

[Hor96] K. Hori, *"Global aspects of gauged Wess–Zumino–Witten models"*, Commun. Math. Phys. **182** (1996), 1–32.

[HS89] C.M. Hull and B. Spence, *"The gauged nonlinear sigma model with Wess–Zumino term"*, Phys. Lett. **B232** (1989), 204–210.

[JJMO90] I. Jack, D.R.T. Jones, N. Mohammedi, and H. Osborn, *"Gauging the general σ-model with a Wess–Zumino term"*, Nucl. Phys. **B332** (1990), 359.
[JK06] J.-H. Jureit and T. Krajewski, *"Quasi-quantum groups from Kalb–Ramond fields and magnetic amplitudes for string orbifolds"*, arXiv preprint: hep-th/0612105.
[Kij73] J. Kijowski, *"A finite-dimensional canonical formalism in the classical field theory"*, Commun. Math. Phys. **30** (1973), 99–128.
[Kij74] ———, *"Multiphase spaces and gauge in the calculus of variations"*, Bull. Acad. Sc. Polon. **22** (1974), 1219–1225.
[KS76] J. Kijowski and W. Szczyrba, *"A canonical structure for classical field theories"*, Commun. Math. Phys. **46** (1976), 183–206.
[KŠ97] C. Klimčík and P. Ševera, *"Open strings and D-branes in WZNW model"*, Nucl.Phys. **B488** (1997), 653–676.
[KT79] J. Kijowski and W.M. Tulczyjew, *A Symplectic Framework For Field Theories*, Lecture Notes in Physics, vol. 107, Springer, 1979.
[LWX98] Z.-J. Liu, A. Weinstein, and P. Xu, *"Manin triples for Lie bialgebroids"*, J. Diff. Geom. **45** (1998), 547–574.
[Mac87] K. MacKenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, 1987.
[ML75] S. Mac Lane, *Homology*, Grundlehren der mathematischen Wissenschaften, vol. 114, Springer, 1975.
[MM03] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, 2003.
[Moe91] I. Moerdijk, *"Classifying toposes and foliations"*, Ann. Inst. Fourier **41** (1991), 189–209.
[Mur96] M.K. Murray, *"Bundle gerbes"*, J. Lond. Math. Soc. **54** (1996), 403–416.
[Ram04] S. Ramanan, *Global Calculus*, Graduate Studies in Mathematics, vol. 65, American Mathematical Society, 2004.
[Ros04a] C.A. Rossi, *"Principal bundles with groupoid structure: local vs. global theory and nonabelian Čech cohomology"*, arXiv preprint: math/0404449 [math.DG].
[Ros04b] ———, *"The division map of principal bundles with groupoid structure and generalized gauge transformations"*, arXiv preprint: math/0401182 [math.DG].
[RS08] A. Recknagel and R.R. Suszek, *"Non-commutative Weitzenböck geometry, gerbe modules, and WZW branes"*, JHEP **0802** (2008), 089.
[RS09] I. Runkel and R.R. Suszek, *"Gerbe-holonomy for surfaces with defect networks"*, Adv. Theor. Math. Phys. **13** (2009), 1137–1219.
[RS11] ———, *"Affine* su(2) *fusion rules from gerbe 2-isomorphisms"*, J. Geom. Phys. **61** (2011), 1527–1552.
[RS12] ———, *"Maximally symmetric defects with junctions in the classical WZW model"*, in writing.
[SSW07] U. Schreiber, C. Schweigert, and K. Waldorf, *"Unoriented WZW models and holonomy of bundle gerbes"*, Commun. Math. Phys. **274** (2007), 31–64.
[Ste00] D. Stevenson, *"The Geometry of Bundle Gerbes"*, Ph.D. thesis, The University of Adelaide, Australia, 2000.
[Sus11] R.R. Suszek, *"Defects, dualities and the geometry of strings via gerbes. I. Dualities and state fusion through defects"*, Hamburger Beiträge zur Mathematik Nr. 360 (2011) [arXiv preprint: 1101.1126 [hep-th]].
[ŠW01] P. Ševera and A. Weinstein, *"Poisson geometry with a 3-form background"*, Prog. Theor. Phys. Suppl. **144** (2001), 145–154.
[SW09] U. Schreiber and K. Waldorf, *"Parallel transport and functors"*, J. Homotopy Relat. Struct. **4** (2009), 187–244.
[SY89] A.N. Schellekens and S. Yankielowicz, *"Extended chiral algebras and modular invariant partition functions"*, Nucl. Phys. **B327** (1989), 673–703.
[SY90] ———, *"Simple currents, modular invariants and fixed points"*, Int. J. Mod. Phys. **A5** (1990), 2903–2952.
[Szc76] W. Szczyrba, *"A symplectic structure on the set of Einstein metrics. A canonical formalism for general relativity"*, Commun. Math. Phys. **51** (1976), 163–182.
[VC92] A.M. Vinogradov and A. Cabras, *"Extensions of the Poisson bracket to differential forms and multi-vector fields"*, J. Geom. Phys. **9** (1992), 75–100.
[Vin90] A.M. Vinogradov, *"Unification of Schouten–Nijenhuis and Frölicher–Nijenhuis brackets, cohomology and super-differential operators"*, Sov. Mat. Zametki **47** (1990), 138–140.
[Wal07] K. Waldorf, *"More morphisms between bundle gerbes"*, Theory Appl. Categories **18** (2007), 240–273.
[Wit92] E. Witten, *"On holomorphic factorization of WZW and coset models"*, Commun. Math. Phys. **144** (1992), 189–212.
[Wu93] S.-Y. Wu, *"Cohomological obstructions to the equivariant extension of closed invariant forms"*, J. Geom. Phys. **10** (1993), 381–392.

Address: Katedra Metod Matematycznych Fizyki, Wydział Fizyki Uniwersytetu Warszawskiego, ul. Hoża 74, PL-00-682 Warszawa, Poland
E-mail address: `suszek@fuw.edu.pl`